Exam I

1. Give an example of three functions f, g, h such that $f \circ (g+h) \neq f \circ g + f \circ h$

   Proof. Let $f(x) = 1, g(x) = 1$ and $h(x) = 1$. $\square$

2. Find the largest natural number m such that $n^3 - n$ is divisible by m for all $n \in \mathbb{N}$. Prove your assertion.

   Proof. Notice that $n^3 - n = (n-1)n(n+1)$ is the product of three consecutive numbers, hence divisible by 6. We claim $m = 6$. Indeed, if $n = 1$ then $n^3 - n = 0$ which is divisible by 6. Suppose $n^3 - n$ is divisible by 6 then $(n+1)^3 - (n+1) = (n^3 - n) + 3n(n+1)$ is also divisible by 6, hence by induction 6 divides all the numbers of form $n^3 - n$, since 6 itself is one of those numbers, it is the maximum divisor. $\square$

3. Show that the set $P = \{ n \in \mathbb{N} ; n \text{ is prime} \}$ is infinite.

   Proof. Suppose $P = \{p_1, p_2, \ldots, p_m\}$ finite. Then the number $p := p_1 \cdot p_2 \cdot \ldots \cdot p_m + 1$ is not divisible by any of the p_i, hence p is itself prime, a contradiction since $p \neq p_i$ for every $i \in \mathbb{N}$. $\square$

4. Given an example of $X_1 \supseteq X_2 \supseteq X_3 \supseteq \ldots$, an infinite sequence of nested **infinite** subsets such that
$$\bigcap_{i=1}^{\infty} X_i = \emptyset$$

   Proof. $X_n = \{n, n+1, n+2, \ldots\}$ or $X_n = (0, \frac{1}{n})$. $\square$

5. Show that the set of all finite subsets of $\mathbb{N}$ is countable.

   Proof. Let $X = \{A \subset \mathbb{N}; A \text{ is finite}\}$ and $X_i = \{A \subset \mathbb{N}; |A| = i\}$. Then
$$X = \bigcup_{i=1}^{\infty} X_i$$
   It's enough to show that X_i is countable for each $i \in \mathbb{N}$. Consider the injective function $f_i : X_i \to \mathbb{N}^i$, that associates to each subset A its elements in $\mathbb{N}^i$. This function is clearly injective. Since $\mathbb{N}^i$ is countable, the result follows. $\square$

6. **Extra.** Prove the induction principle using the well-ordering principle. *(Try proof by contradiction)*

   Proof. Suppose the principle of well-ordering is true and $X \subseteq \mathbb{N}$ has the property that $1 \in X$ and $n \in X \Rightarrow n+1 \in X$. Suppose that $X \neq \mathbb{N}$, by the principle of well-ordering $\mathbb{N} - X$ has a minimum element, say m. Since $m \neq 1$, m is the successor of an element, say a, i.e. m=a+1, by minimality of m we must have $a \in X$, a contradiction since $a + 1 = m \notin X$. $\square$