Exam II

Choose 5 questions below.

1. Prove that if a increasing sequence x_n has a convergent subsequence x_{n_k} then x_n is convergent. *Hint: Remember that every bounded monotone sequence converges.*

   Solution. It suffices to show that x_n is bounded. Since x_{n_k} converges, it is bounded, say $x_{n_k} \leq M$. Now, for any $n \in \mathbb{N}$, we can find $n_k > n$, but then $x_n < x_{n_k} \leq M$. It follows that x_n is bounded by M as well.

2. Show that $\sum \frac{\ln n}{n^3}$ converges. *Hint: Remember that* $\ln n < n$.

   Solution. Notice that $\frac{\ln n}{n^3} < \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{\ln n}{n^3}$ converges by comparison.

3. Let x_n be a sequence defined by

$$x_n = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \frac{1}{n}, & \text{if } n \text{ is even.} \end{cases}$$

   Find all accumulation points of x_n. Does x_n converge?

   Solution. Notice that $x_{2n-1} = 2n - 1$ diverges, so x_n has a divergent subsequence, hence it can't converge. If a is an accumulation point, it is the limit of a subsequence, say x_{n_k}, of x_n. The indexes n_k have to be necessarily even, otherwise x_{n_k} doesn't converge. But $x_{2n-1} \to 0$, hence $x_{n_k} \to 0$. It follows that $a = 0$, so 0 is the only accumulation point.

4. Give an example of a convergent series $\sum a_n$ and a bounded sequence b_n such that $\sum a_n b_n$ is divergent. *Hint: Remember for example that* $\sum \frac{(-1)^n}{n}$ *converges.*

   Solution. $a_n = \frac{(-1)^n}{n}$ and $b_n = (-1)^n$.

5. Let x_n be a sequence of positive numbers such that $\lim x_n = a$. Prove that

$$\lim \sqrt[n]{x_1 x_2 x_3 \ldots x_n} = a.$$

   Hint: Remember that if $\frac{y_{n+1}}{y_n} \to c$ *then* $\sqrt[n]{y_n} \to c$.

   Solution. Let $y_n = x_1 x_2 x_3 \ldots x_n$ then $\frac{y_{n+1}}{y_n} = x_{n+1} \to a$, hence $\sqrt[n]{x_1 x_2 x_3 \ldots x_n} = \sqrt[n]{y_n} \to a$.

6. Let x_n be defined by $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + x_n}$$

Show that x_n converges and find its limit.

Solution. We claim using induction that x_n is increasing. Start by noticing that $\sqrt{2} = x_1 < x_2 = \sqrt{2 + x_1}$. Suppose $x_n < x_{n+1}$. Then

$$2 + x_n < 2 + x_{n+1} \Rightarrow \sqrt{2 + x_n} < \sqrt{2 + x_{n+1}} \Rightarrow x_{n+1} < x_{n+2}$$

Now, suppose x_n converges to L, then taking the limit on $x_{n+1} = \sqrt{2 + x_n}$ we obtain $L = \sqrt{2 + L} \Rightarrow L^2 - L - 2 = 0$. It follows that $L = -1$ or $L = 2$, since $x_1 = \sqrt{2}$ and x_n increases, $L = 2$ is only possibility. We now prove that x_n converges. It suffices to show that x_n is bounded (since it's monotone). We claim by induction that $x_n < 2$ for every n. The case $n = 1$ is clear, suppose $x_n < 2$ then $x_n + 2 < 2 + 2 \Rightarrow \sqrt{x_n + 2} < \sqrt{4} \Rightarrow x_{n+1} < 2$.

7. Show that the series $\sum \frac{1}{n^n}$ converges.

Solution. Notice that $\sqrt[n]{\frac{1}{n^n}} = \frac{1}{n} \to 0 < 1$ as $n \to +\infty$. Hence the series converges by the root test.

8. Let $a > 0$ and $a \neq e$. Find

$$\lim \frac{a^n n!}{n^n}.$$

Hint: Remember that if $\frac{x_{n+1}}{x_n} \to c < 1$ *then* $x_n \to 0$.

Solution. Let $x_n := \frac{a^n n!}{n^n}$ then $\lim \frac{x_{n+1}}{x_n} = \frac{a}{e}$, if $a < e$ then $\lim \frac{x_{n+1}}{x_n} < 1$ and it follows that $x_n \to 0$. If $a > e$ then $\lim \frac{x_{n+1}}{x_n} > 1$, hence $x_n \to +\infty$.