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Real Analysis

Functions of a real variable

Springer Nature

To my wonderful family.

Preface

This book is intended as a first rigorous introduction to Real Analysis, with a focus on functions of a real variable. It is written for undergraduate students in mathematics, but it may also serve as a reference for anyone seeking a clear and rigorous introduction to the subject. The exposition is self contained and only assumes a basic background in elementary mathematics.

A distinctive feature of the book is its balance between theory and practice. Many theorems are followed by worked examples and exercises of varying difficulty, encouraging the reader to actively engage with the material and develop problem-solving skills. The style is direct and concise, yet aims to be accessible; historical notes are included to give context and show the evolution of ideas in mathematics.

The text begins with set theory and the real number system, then moves through sequences, limits, continuity, derivatives, and integrals, ending with more advanced topics such as uniform convergence and power series. Each chapter blends theory with examples, and includes exercises to reinforce understanding and encourage active learning. My goal is to make the subject clear, precise, and engaging, while maintaining the rigor essential to mathematical analysis.

I hope readers will not only master the techniques but also appreciate the beauty and logic that make Real Analysis a central part of mathematics.

San Antonio, TX, August 2025
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Contents

Chapter 1
Naive Set Theory

In this chapter, we will introduce the notions of sets and functions. These are fundamental notions that will be used extensively throughout the remainder of the text. In fact, the main goal of this textbook is to study functions of a real variable defined on subsets of the real number field. We will begin this study with basic definitions related to sets, then the natural numbers will be introduced, and finally, we will compare the naturals with various types of sets.

1.1 Sets

A **set** X is a collection of objects, also called the *elements* of the set. If 'a' is an element of X, we write $a \in X$. On the other hand, if 'a' isn't an element of X, we write $a \notin X$.

A set X is *well defined* when there is a rule that allows one to precisely determine if an arbitrary element 'a' is or is not an element of X.

Example 1.1. The set X of all right triangles is well-defined. Indeed, given any object 'a', if 'a' is not a triangle or does not have a right angle then $a \notin X$. If 'a' is a right triangle, then $a \in X$.

Example 1.2. * The set X of all sets that are not members of themselves is not well-defined, because we cannot say whether X, as an element itself, is or is not an element of X. If X is an element of X, then by the definition of X, X is not in X. Conversely, if X is not in X, then X is in X.

Usually one uses the notation

$$X = \{a, b, c, \ldots\}$$

* This example is known as **Russell's Paradox**. After its publication in 1901, Ernst Zermelo proposed an axiomatic theory of sets where the notion of a set is made more precise.

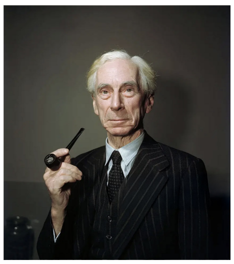
to represent the set X whose elements are a,b,c, and so on. If a set has no elements, we denote it by $\emptyset$, and call it the **empty set**.

The set of *natural numbers* $1, 2, 3, \ldots$ will be denoted by

$$\mathbb{N} = \{1, 2, 3, \ \ldots\}$$

The set of *integers* will be denoted by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ \ldots\}$$



■ Bertrand Russell was a British philosopher and mathematician. He had a significant influence on the foundations of mathematics, especially in set theory.

The set of *rational numbers*, that is, fractions $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$, will be denoted by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, \ b \neq 0 \right\}.$$

In chapter 2, we will formally define the set of *real numbers*, denoted by $\mathbb{R}$. For now, we consider the set $\mathbb{R}$ of real numbers to consist of all numbers that have a decimal representation.

The vast majority of sets in mathematics are not defined by specifying their elements one by one. What usually happens is that a set is defined by some property its elements satisfy; i.e., if a has property P, then $a \in X$, whereas if a does not have property P, then $a \notin X$. One writes

$$X = \{a \mid a \text{ has property } P\} \text{ or } X = \{a \,;\, a \text{ has property } P\}$$

both notations will be used in the text.

Example 1.3. The set

$$X = \{a \in \mathbb{N} \mid a > 10\},$$

consists of all natural numbers greater than 10, namely, $X = \{11, 12, 13, \ldots\}$.

Given two sets A and B, one says that A is a **subset** of B, or that A is *included* in B (i.e., B *contains* A), denoted by $A \subseteq B$, if every element of A is also an element of B.

When one writes $X \subseteq Y$, it is possible that $X = Y$. In the case where $X \neq Y$, we say that X is a *proper subset*. The notation $X \subsetneq Y$ is sometimes used to indicate that X is a proper subset of Y.

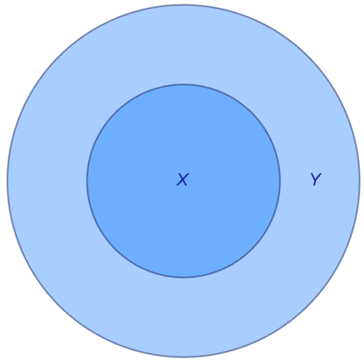


Fig. 1.1: Picture of a set X as a subset of Y.

Example 1.4. We have the obvious inclusion of sets:

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Example 1.5. Let X be the set of all squares and Y be the set of all rectangles. Then $X \subseteq Y$, since every square is a rectangle.

Notice that to write $a \in X$ is equivalent to say $\{a\} \subseteq X$. Also, by definition, it's always true that $\emptyset \subseteq X$ for every set X.

It's easy to see that the inclusion of sets has the following properties:

1. *Reflexive*: $X \subseteq X$ for every set X;
2. *Antisymmetric*: if $X \subseteq Y$ and $Y \subseteq X$, then $X = Y$;
3. *Transitive*: if $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$.

It follows that two sets X and Y are the same if and only if $X \subseteq Y$ and $Y \subseteq X$, that is, they have the same elements.

The *power set* of the set X, denoted by$\mathcal{P}(X)$, is defined as the set

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

The set $\mathcal{P}(X)$ denotes the collection of all subsets of the set X. In particular, it is never empty, as it always contains at least the empty set $\emptyset$ and the set X itself.

Example 1.6. Let $X = \{1, 2, 3\}$ then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Notice that, by using the Fundamental Counting Principle, any set with n elements has 2^n subsets. Therefore, the number of elements of $\mathcal{P}(X)$ is 2^n.

Given two sets X and Y, one can build many other sets. For example, the **union** of X and Y, denoted by $X \cup Y$, is the set of elements that are in X or Y. More precisely:

$$X \cup Y = \{ a \mid a \in X \text{ or } a \in Y \}.$$

Similarly, the **intersection** of X and Y, denoted by $X \cap Y$ is the set of elements that are common to both X and Y:

$$X \cap Y = \{ a \mid a \in X \text{ and } a \in Y \}.$$

If $X \cap Y = \emptyset$, then X and Y are said to be *disjoint*.

Example 1.7. Let $X = \{a \in \mathbb{N} \mid a \leq 100\}$ and $Y = \{a \in \mathbb{N} \mid a > 50\}$ then

$$X \cup Y = \mathbb{N} \text{ and } X \cap Y = \{a \in \mathbb{N} \mid 50 < a \leq 100\}$$

Example 1.8. Consider the sets

$$X = \{1, 2, \{3\}\} \text{ and } Y = \{\{1, 2\}, 3\}$$

Then $X \cap Y = \emptyset$, $X \cup Y = \{1, 2, 3, \{1, 2\}, \{3\}\}$.

Example 1.9. The sets $X = \{a \in \mathbb{N} \mid a > 1\}$ and $Y = \{a \in \mathbb{N} \mid a < 2\}$ are disjoint, since there is no natural number between 1 and 2.

The **difference** between X and Y, denoted by $X - Y$ is the set of elements that are in X but not in Y, more precisely:

$$X - Y = \{ a \mid a \in X \text{ and } a \notin Y \}.$$

Given an inclusion of sets $X \subseteq Y$, the **complement** of X in Y is the set $Y - X$. The notation X^c is sometimes used when there is no ambiguity about the set Y.

Example 1.10. Let $A, B \subseteq X$. Then $A \cap B = \emptyset$ if and only if $A \subseteq B^c$. Indeed, if $A \cap B = \emptyset$ then $x \in A \Rightarrow x \notin B$, hence $A \subseteq B^c$. Conversely, suppose $A \subseteq B^c$. Assume, by contradiction, that $A \cap B \neq \emptyset$, and let $x \in A \cap B$. Then $x \in A$ and $x \in B$, in particular $x \notin B^c$, contradicting the assumption that $A \subseteq B^c$. Therefore, $A \cap B = \emptyset$.

Example 1.11. Consider the sets $X = \{a \in \mathbb{N} \mid a \text{ is even}\}$ and $Y = \mathbb{N}$. Then $X \subseteq Y$ and $X^c = \{a \in \mathbb{N} \mid a \text{ is odd}\}$. A similar example is $\mathbb{Q} \subseteq \mathbb{R}$, and $\mathbb{Q}^c$ is the set of all irrational numbers—those numbers that are not rational (i.e., not expressible as fractions).

Example 1.12. Observe that $\mathbb{Z} - \mathbb{N}$ is the set of all non-positive integers, including zero. Similarly, the set $\mathbb{Q} - \mathbb{Z}$ consists of all rational numbers that are not integers. For example, $\frac{1}{2} \in \mathbb{Q} - \mathbb{Z}$.

Theorem 1.13. *Given sets A, B, C, D the following properties are true:*

1. $A \cup \emptyset = A$; $A \cap \emptyset = \emptyset$
2. $A \cup A = A$; $A \cap A = A$
3. $A \cup B = B \cup A$; $A \cap B = B \cap A$
4. $A \cup (B \cup C) = (A \cup B) \cup C$; $A \cap (B \cap C) = (A \cap B) \cap C$

5. *$A \cup B = A \Leftrightarrow B \subseteq A$; $A \cap B = A \Leftrightarrow A \subseteq B$*
6. *if $A \subseteq B$ and $C \subseteq D$ then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$*
7. *$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$*
8. *$(A^c)^c = A$*
9. *$(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$*

Proof. We prove the last property, $(A \cup B)^c = A^c \cap B^c$. The others are trivial or can be proved in a similar way.

We show that $(A \cup B)^c \subseteq A^c \cap B^c$. Let $a \in (A \cup B)^c$. Then $a \notin A \cup B$; in particular, $a \notin A$ and $a \notin B$. Hence, $a \in A^c \cap B^c$.

Conversely, take $a \in A^c \cap B^c$. Then $a \notin A$ and $a \notin B$, so $a \notin A \cup B$, and it follows that $a \in (A \cup B)^c$. □

An *ordered pair* (a, b) is formed by two objects a and b, such that for any other such pair (c, d):

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

The elements a and b are called *coordinates* of (a, b): a is the first coordinate, and b the second one.

Remark.

An ordered pair is not the same as a set; that is, $(a, b) \neq \{a, b\}$. Notice that $\{a, b\} = \{b, a\}$, but in general, $(a, b) \neq (b, a)$. The **cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{ (x, y) \mid x \in X \text{ and } y \in Y \}.$$

Example 1.14. The set $\mathbb{N} \times \mathbb{N}$ consists of all ordered pairs (a, b) whose coordinates are natural numbers.

Example 1.15. The sets $\mathbb{Z} \times \{0\}$ and $\mathbb{Z} \times \{1\}$ are disjoint. Additionally, $\mathbb{N} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{N} \subseteq \mathbb{Z} \times \mathbb{Q}$.

Example 1.16. Consider the sets $X = \{1, 2, 3\}$ and $Y = \{0, 1\}$, then

$$X \times Y = \{ (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (3, 1) \}.$$

Example 1.17. The definition of the cartesian product can be generalized to more than two sets. For example, given sets X,Y,Z, one may define $X \times Y \times Z$ as the collection of all triples (a, b, c), such that $a \in X, b \in Y, c \in Z$. In other words:

$$X \times Y \times Z = \{ (x, y, z) \mid x \in X, y \in Y \text{ and } z \in Z \}.$$

■ Gottfried Leibniz was a German mathematician who, alongside Sir Isaac Newton, is credited with the creation of calculus. One of the earliest definitions of a function is also attributed to him.

1.2 Functions

A **function** $f : X \longrightarrow Y$ consists of three components: a set X, *the domain*; a set Y, the *codomain*; and a rule that associates each element $a \in X$ with a unique element $f(a) \in Y$. The value $f(a)$ is called the *value* of f at a, or the image of a under f.

Another common notation to denote a function is $x \mapsto f(x)$. In this case the domain and codomain can be identified by the context.

Example 1.18. The function $f : \mathbb{N} \longrightarrow \mathbb{N}$ given by $f(n) = n + 1$ is called the successor function.

Example 1.19. Let X be the set of all triangles. One can define a function $f : X \longrightarrow \mathbb{R}$ by $f(x) =$ area of x.

Example 1.20. The correspondence that associates to each real number x the set of all y satisfying $y^2 = x$ is not a function, because any $x \neq 0$ is associated with two values, namely $\pm\sqrt{x}$. In order to be a function, each x must be associated with exactly one value $y = f(x)$.

Example 1.21. A *family of sets* is a function $X : \Lambda \longrightarrow Y$ such that $X(n)$ (also denoted X_n) is a set for every $n \in \Lambda$. The domain Λ is called the index set, and when $\Lambda = \mathbb{N}$, we call $X : \mathbb{N} \longrightarrow Y$ a sequence of sets. It's customary to denote a family of sets by

$$\{X_n\}_{n\in\Lambda}$$

For example, the function $X : \mathbb{N} \longrightarrow \mathcal{P}(\mathbb{N})$ given by

$$X_n = \{m; m \geq n\}$$

defines a sequence of (sub)sets.

The graph of a function $f : X \longrightarrow Y$ is the subset of $X \times Y$ defined by

$$\Gamma(f) = \{\, (x, f(x)) \mid x \in X \,\}.$$

Example 1.22. The rule $x \mapsto e^{-x^2}$ defines a function $f : \mathbb{R} \longrightarrow \mathbb{R}$. This function is extensively used in probability theory due to its unique properties. It is commonly referred to as the *Gaussian function*.

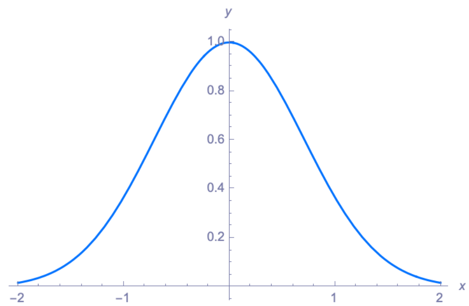


Fig. 1.2: The graph of the function $f(x) = e^{-x^2}$.

A function $f : X \longrightarrow Y$ is said to be *injective* (or one-to-one) if, for every $x, y \in X$, whenever $f(x) = f(y)$, it follows that $x = y$. Similarly, a function $f : X \longrightarrow Y$ is said to be *surjective* (or onto) if, for every $y \in Y$, there exists an $x \in X$ such that $f(x) = y$. Finally, a function $f : X \longrightarrow Y$ is *bijective* (or a bijection) if it is both injective and surjective.

Example 1.23. The function given by $f(x) = x^3$ is injective.

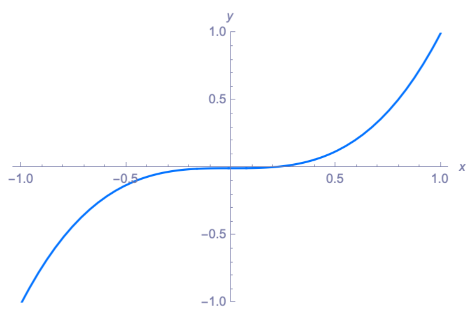


Fig. 1.3: The graph of the function $f(x) = x^3$.

Example 1.24. Given a set $X \subseteq \mathbb{R}$, we denote by $\max X$, the largest element of X. An example of a function that is not injective is given by the floor function $\lfloor x \rfloor = \max\{\, n \in \mathbb{Z} \mid n \leq x \,\}$.

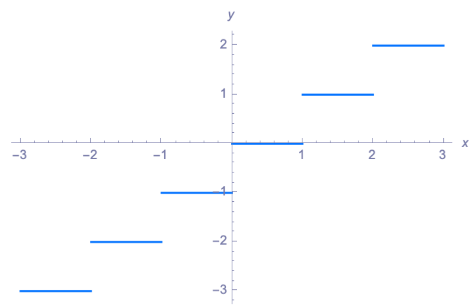


Fig. 1.4: The graph of the function $f(x) = \lfloor x \rfloor$.

Example 1.25. The function $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}$ given by $f(x) = \sin x$ is a bijection.

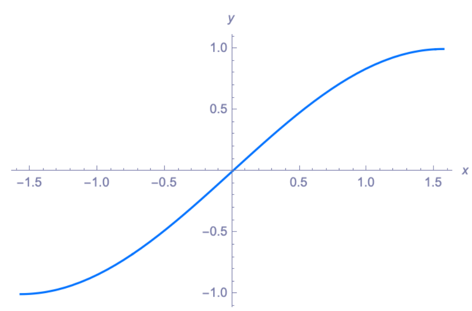


Fig. 1.5: The graph of the function $f(x) = \sin x$.

Given a function $f : X \longrightarrow Y$, the *image of a set* $A \subseteq X$ is defined by

$$f(A) = \{\, y \in Y \mid y = f(a),\ a \in A \,\}.$$

Conversely, the *inverse image of a set* (sometimes called *pre-image*) $B \subseteq Y$ is defined by

$$f^{-1}(B) = \{\, x \in X \mid f(x) \in B \,\}.$$

Theorem 1.26. *Given $f : X \longrightarrow Y$ and subsets $A, B \subseteq X$, we have:*

1. $f(A \cup B) = f(A) \cup f(B)$; $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
2. $f(A \cap B) \subseteq f(A) \cap f(B)$; $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
3. *if* $A \subseteq B$ *then* $f(A) \subseteq f(B)$ *and* $f^{-1}(A) \subseteq f^{-1}(B)$
4. $f(\emptyset) = \emptyset$; $f^{-1}(\emptyset) = \emptyset$
5. $f^{-1}(Y) = X$
6. $f^{-1}(A^c) = (f^{-1}(A))^c$

Proof. These properties follow directly from the definitions. We prove the last one for clarity of exposition; the others can be established in a similar manner.

The last item follows from the following chain of equivalences:

$$x \in f^{-1}(A^c) \iff f(x) \in A^c \iff f(x) \notin A \iff x \notin f^{-1}(A)$$

□

Example 1.27. Consider the function $f : \mathbb{Q} \to \mathbb{Z}$ defined by $\frac{a}{b} \mapsto a \cdot b$, where $gcd(a, b) = 1$. Then, for all $n \in \mathbb{Z}$:

$$f^{-1}(\{n\}) = \{n, \frac{1}{n}\} \text{ if } n \neq \pm 1, \quad \text{and } f^{-1}(\{\pm 1\}) = \{\pm 1\}$$

Given two functions $f : X \to Y$ and $g : Y \to Z$, the *composition* $g \circ f$ of g and f is defined as the function:

$$(g \circ f)(x) = g(f(x))$$

Example 1.28. The composition of the functions $g(x) = \sin x$ and $f(x) = e^x$ is the function $(g \circ f)(x) = \sin e^x$ depicted below.

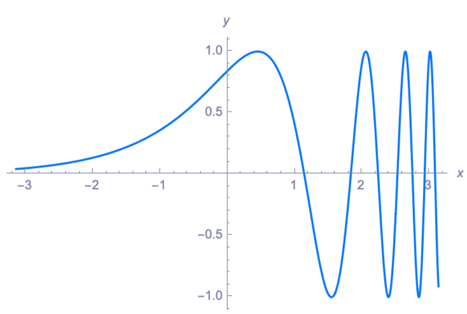


Fig. 1.6: The graph of the function $f(x) = \sin e^x$.

Example 1.29. The function $f(x) = \ln \sqrt{x}$ is the composition of $\ln x$ and $\sqrt{x}$. Similarly, the function $g(x) = \sin \frac{1}{x^2}$ is the composition of $\sin x$ and $\frac{1}{x^2}$. The function $h(x) = f(x) + xg(x)$ is depicted below.

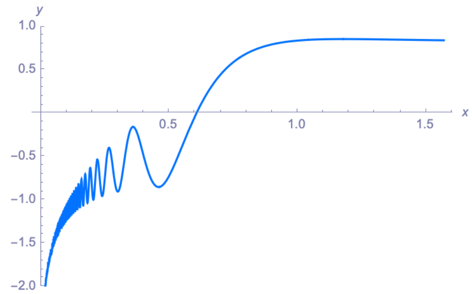


Fig. 1.7: The graph of the function $f(x) = \ln\sqrt{x} + x\sin\frac{1}{x^2}$.

Given a function $f : X \longrightarrow Y$ and a subset $A \subseteq X$, *the restriction* of f to A, denoted by $f|_A : A \longrightarrow Y$, is defined by $f|_A(x) = f(x)$. Similarly, if $X \subseteq Z$, *an extension* of f to Z is any function $g : Z \longrightarrow Y$ such that $g|_X(x) = f(x)$.

Given functions $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$, the function g is called a *left inverse* of f if

$$(g \circ f)(x) = x \quad \text{for all } x \in X.$$

Similarly, the function $g(x)$ is called *right-inverse* of $f(x)$ if

$$(f \circ g)(x) = x \quad \text{for all } x \in Y.$$

Finally, if there is a function $f^{-1}(x)$ such that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x,$$

then $f^{-1}(x)$ is called the *inverse* of $f(x)$. Note that any inverse, if exists, is unique. If $g(x)$ and $h(x)$ are both inverses of $f(x)$ then

$$g(x) = g(f(h(x))) = (g \circ f)(h(x)) = h(x).$$

Theorem 1.30. *A function $f : X \longrightarrow Y$ has an inverse $f^{-1} : Y \longrightarrow X \Leftrightarrow f$ is bijective.*

Proof. Suppose f has an inverse f^{-1}, and let $f(x) = f(y)$ for some $x, y \in X$. Applying f^{-1} to both sides, we get

$$f^{-1}(f(x)) = f^{-1}(f(y)) \Rightarrow x = y,$$

so f is injective.

To show surjectivity, let $y \in Y$. Since $f^{-1} : Y \longrightarrow X$ is defined, set $x = f^{-1}(y)$. Then

$$f(x) = f(f^{-1}(y)) = y,$$

so every $y \in Y$ has a preimage in X, and thus f is surjective.

Conversely, suppose f is bijective. For each $y \in Y$, since f is surjective, there exists $x \in X$ such that $f(x) = y$. Define $f^{-1}(y) = x$. Because f is also injective, this definition is unambiguous. Then,

$$(f \circ f^{-1})(y) = f(f^{-1}(y)) = y \quad \text{and} \quad (f^{-1} \circ f)(x) = f^{-1}(f(x)) = x,$$

so f^{-1} is indeed the inverse of f. □

Example 1.31. Consider the function $f : (0, +\infty) \to (0, +\infty)$ defined by $f(x) = \frac{1}{x}$. Then f is its own inverse; that is,

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = x.$$

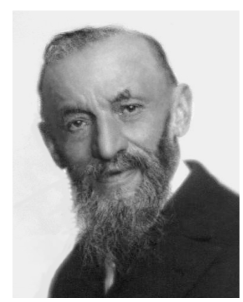
Similarly, the function $g(x) = \ln\left(\frac{e^x+1}{e^x-1}\right)$ is its own inverse. More generally, any function whose graph is symmetric with respect to the line $y = x$ is its own inverse.

1.3 The natural numbers $\mathbb{N}$

The natural numbers are built axiomatically. We begin with a set $\mathbb{N}$, whose elements are called *natural numbers*, and a function $s : \mathbb{N} \to \mathbb{N}$, called the *successor function*. For any $n \in \mathbb{N}$, $s(n)$ is called the successor of n.

The function $s(n)$ satisfies the following axioms, known as *Peano's axioms*:

Axiom 1. The function $s(n)$ is injective; that is, every number has a unique successor.
Axiom 2. The set $\mathbb{N} \setminus s(\mathbb{N})$ has exactly one element, denoted by 1; in other words, every number has a successor, and 1 is not the successor of any number.
Axiom 3. *(Principle of induction)* Let $X \subseteq \mathbb{N}$ be a subset such that $1 \in X$, and whenever $n \in X$, it follows that $s(n) \in X$. Then $X = \mathbb{N}$.



■ Giuseppe Peano was an Italian mathematician. The standard axiomatization of the natural numbers is named the Peano axioms in his honor.

Whenever Axiom 3 is used to prove a result, the result is said to be proved by induction.

Theorem 1.32. *For any $n \in \mathbb{N}$, we have $s(n) \neq n$.*

Proof. We proceed by induction. Define the set $X \subset \mathbb{N}$ by

$$X = \{\, n \in \mathbb{N} \mid s(n) \neq n \,\}.$$

By Axiom 2, we have $1 \in X$. Now, assume $n \in X$; that is, $s(n) \neq n$. Then by Axiom 1, it follows that $s(s(n)) \neq s(n)$, hence $s(n) \in X$. Therefore, by Axiom 3, we conclude that $X = \mathbb{N}$, and the result follows. □

Given a function $f : X \to X$, its powers f^n are defined inductively. Specifically, set $f^1 = f$, and for each $n \in \mathbb{N}$, define

$$f^{s(n)} = f \circ f^n.$$

In particular, if we define $2 = s(1)$, $3 = s(2)$, and so on, then we obtain:

$$f^2 = f \circ f, \quad f^3 = f \circ f \circ f, \quad \ldots$$

Now, given two natural numbers $m, n \in \mathbb{N}$, their sum $m + n \in \mathbb{N}$ is defined by

$$m + n = s^n(m),$$

where s^n denotes the n-fold composition of the successor function s.

It follows that $m + 1 = s(m)$, and more generally,

$$m + s(n) = s(m + n).$$

In particular, we have the recursive identity:

$$m + (n + 1) = (m + n) + 1.$$

Lemma 1.33. *For all $n, q \in \mathbb{N}$, we have $n + q \neq n$.*

Proof. We proceed by induction on n, for a fixed $q \in \mathbb{N}$. The result is true when $n = 1$ by Theorem 1.32. Assume the statement holds for some $n \in \mathbb{N}$; that is,

$$n + q \neq n.$$

We must show that $(n + 1) + q \neq n + 1$.

Note that

$$(n + 1) + q = s(n) + q = s(n + q),$$

and by the inductive hypothesis, $n + q \neq n$. Since the successor function s is injective, it follows that

$$s(n + q) \neq s(n),$$

i.e.,

$$(n + 1) + q \neq n + 1.$$

This completes the inductive step. Since $q \in \mathbb{N}$ was arbitrary, the proof of the lemma is complete. □

More generally, the addition of two natural numbers satisfies the following properties.

Theorem 1.34. *For any $m, n, p \in \mathbb{N}$:*

1. *(Associativity) $m + (n + p) = (m + n) + p$;*
2. *(Commutativity) $m + n = n + m$;*
3. *(Cancellation Law) $m + n = m + p \Rightarrow n = p$;*

4. *(Trichotomy) Only one of the following can occur:* $m = n$, *or* $\exists q \in \mathbb{N}$ *such that* $m = n + q$, *or* $\exists r \in \mathbb{N}$ *such that* $n = m + r$.

Proof. The first and second properties are straightforward. Suppose $m + n = m + p$, then

$$s^m(n) = s^m(p),$$

but since s is injective, $s^{m-i}(n) = s^{m-i}(p)$ for each $i = 1, 2, \ldots, n - 1$. This proves 3. To prove 4, suppose $\exists q \in \mathbb{N}$ such that $m = n + q$. Then if $m = n$, we have $n + q = n$, a contradiction by Lemma 1.33 above. Similarly, if there $\exists r \in \mathbb{N}$ such that $n = m + r$ then $m = (m + r) + q$, as before, this is a contradiction. □

The notion of order among natural numbers can be defined in terms of addition. Specifically, we write

$$m < n$$

if there exists $q \in \mathbb{N}$ such that $n = m + q$. In this case, we also write $n > m$.

In particular, for every $m \in \mathbb{N}$,

$$m < s(m),$$

since $s(m) = m + 1$ by definition of the successor function.

We define $m \geq n$ to mean $m > n$ or $m = n$, and similarly for $m \leq n$.

The following corollary is an immediate consequence of Theorem 1.34.

Corollary 1.35. *For any* $m, n, p \in \mathbb{N}$*:*

1. *(Transitivity)* $m < n, n < p \Rightarrow m < p$;
2. *(Trichotomy) Only one of the following can occur:* $m = n$, $m < n$ *or* $m > n$.
3. $m < n \Rightarrow m + p < n + p$.

The multiplication operation $m \cdot n$ is defined in a manner analogous to how addition $m + n$ was defined.

Let $a_m : \mathbb{N} \to \mathbb{N}$ be the "add-m" function, defined by $a_m(n) = n + m$. Then, the product of two natural numbers $m \cdot n$ is defined recursively as follows:

$$m \cdot 1 := m,$$
$$m \cdot (n + 1) := (a_m)^n(m),$$

where $(a_m)^n$ denotes the n-fold composition of the function a_m.

For example,

$$m \cdot 2 = a_m(m) = m + m, \quad m \cdot 3 = (a_m)^2(m) = m + m + m, \quad \text{and so on.}$$

It follows from this definition that multiplication satisfies the recursive relation:

$$m \cdot (n + 1) = m \cdot n + m.$$

More generally, using the same ideas as in the proof of Theorem 1.34, one can establish the properties below. The proof is left as an exercise for the reader.

Theorem 1.36. *For any $m, n, p \in \mathbb{N}$:*

$m \cdot (n \cdot p) = (m \cdot n) \cdot p;$
$m \cdot n = n \cdot m;$
$m \cdot n = p \cdot n \Rightarrow m = p;$
$m \cdot (n + p) := m \cdot n + m \cdot p;$
$m < n \Rightarrow m \cdot p < n \cdot p.$

1.4 Well-ordering principle

Let $X \subseteq \mathbb{N}$. A number $m \in X$ is called the **minimum element** of X, denoted

$$m = \min X,$$

if $m \leq n$ for every $n \in X$.

The **maximum element** is defined analogously: $m = \max X$ if $m \geq n$ for all $n \in X$.

Note that not every subset $X \subseteq \mathbb{N}$ has a maximum. In fact, $\mathbb{N}$ itself has no maximum, since $m < m + 1$ for every $m \in \mathbb{N}$.

Example 1.37. The minimum element of the set

$$X = \{n \in \mathbb{N} \mid n^2 + 1 > 50\}$$

is 8, i.e., $\min X = 8$. However, X does not have a maximum, since $n \in X \Rightarrow n + 1 \in X$.

Lemma 1.38. *If $m = \min X$ and $n = \min X$ then $m = n$. A equivalent result is true for the maximum.*

Proof. Since $m \leq p$ for every $p \in X$, $m \leq n$ in particular. Similarly, $n \leq m$ and hence $m = n$. □

Although not every subset of $\mathbb{N}$ has a maximum, every non-empty subset does have a minimum.

Theorem 1.39. *(Well-ordering principle) Let $X \subseteq \mathbb{N}$ be non-empty. Then X has a minimum.*

Proof. If $1 \in X$ then 1 is the minimum, so suppose $1 \notin X$. Let

$$I_n = \{\, m \in \mathbb{N} \,|\, 1 \leq m \leq n \,\},$$

and consider the set

$$L = \{\, n \in \mathbb{N} \,|\, I_n \subseteq X^c \,\}.$$

Since $1 \notin X \Rightarrow 1 \in L$. If $n \in L \Rightarrow n + 1 \in L$, the induction hypothesis would imply $L = \mathbb{N}$, but $L \neq \mathbb{N}$, since $L \subseteq X^c = \mathbb{N} - X$, and $X \neq \emptyset$. We conclude that there is a m_0 such that $m_0 \in L$ and $m_0 + 1 \notin L$. It follows than $m_0 + 1$ is the minimum element of X. □

Corollary 1.40. *(Strong induction) Let $X \subseteq \mathbb{N}$ be a set with the following property:*

$$\forall n \in \mathbb{N}, \text{ if } X \text{ contains all } m < n \Rightarrow n \in X.$$

Then $X = \mathbb{N}$.

Proof. Define $Y = X^c$. The result is equivalent to the statement $Y = \emptyset$. Suppose not, that is, $Y \neq \emptyset$. By the Well-ordering principle, Y has a minimum element, say $p \in Y$. It follows that $p \in X$, a contradiction. □

Example 1.41. Strong induction can be used to prove the **Fundamental Theorem of Arithmetic**, which states that every natural number greater than 1 can be written as a product of prime numbers. (A number p is called **prime** if $p > 1$ and whenever $p = m \cdot n$, then either $m = 1$ or $n = 1$.)

Let $X = \{m \in \mathbb{N} \mid m > 1 \text{ and } m \text{ is a product of primes}\}$, and fix $n \in \mathbb{N}$ with $n > 1$. Suppose that X contains all natural numbers m such that $1 < m < n$.

If n is prime, then $n \in X$. If n is not prime, then $n = p \cdot q$ for some $p, q < n$, and by the inductive hypothesis, both p and q belong to X, so n is a product of primes. In either case, it follows that $n \in X$.

Therefore, by the principle of strong induction, we conclude that $X = \{n \in \mathbb{N} \mid n > 1\}$, i.e., every natural number greater than 1 is a product of primes.

Let X be any set. A common method for defining a function $f : \mathbb{N} \to X$ is **by recurrence** (this is sometimes also referred to as "by induction" or "recursively"). Specifically, one defines $f(1)$, and provides a rule that determines $f(m)$ based on the values of $f(n)$ for all $n < m$.

In principle, more than one function could satisfy such conditions. However, one can easily show that the function defined in this way is unique.

Example 1.42. (Factorial) The factorial function $n \mapsto n!$ can be defined recursively. Define a function $f : \mathbb{N} \to \mathbb{N}$ by setting:

$$f(1) = 1, \quad \text{and} \quad f(n+1) = (n+1) \cdot f(n).$$

Then,

$$f(2) = 2 \cdot 1, \quad f(3) = 3 \cdot 2 \cdot 1, \quad \dots, \quad f(n) = n!.$$

Example 1.43. (Arbitrary sums/products) So far, we have defined expressions such as $m + n$. What about sums involving more terms, such as $m + n + p$ or general expressions like $m_1 + \cdots + m_n$? To define such arbitrary sums (and similarly products), we use induction.

The sum of n terms is defined recursively as:

$$m_1 + \cdots + m_n := (m_1 + \cdots + m_{n-1}) + m_n.$$

Similarly, for products:

$$m_1 \cdot \cdots \cdot m_n := (m_1 \cdot \cdots \cdot m_{n-1}) \cdot m_n.$$

1.5 Finite and Infinite Sets

Throughout this section, I_n denotes the set of natural numbers less than or equal to n:

$$I_n = \{m \in \mathbb{N} \mid 1 \leq m \leq n\}.$$

An arbitrary set X is called **finite** if $X = \emptyset$ or there exists a number $n \in \mathbb{N}$ and a bijection

$$f : I_n \longrightarrow X.$$

In the latter case, we say that X has n elements and write

$$|X| = n.$$

The function f is called a *counting function* for X. By convention, if $X = \emptyset$, then X is said to have zero elements; that is, $|\emptyset| = 0$.

It remains to show that the notion of "number of elements" is well-defined. That is, if there exist bijections $f : I_n \longrightarrow X$ and $g : I_m \longrightarrow X$, then it must follow that $n = m$.

Theorem 1.44. *Let $X \subseteq I_n$. If there is a bijection $f : I_n \longrightarrow X$, then $X = I_n$.*

Proof. The proof is by induction on n. The case $n = 1$ is obvious, suppose the result true for n, the proof follows if one can prove the result for $n + 1$.

Suppose $X \subseteq I_{n+1}$ and there is a bijection $f : I_{n+1} \longrightarrow X$. Let $a = f(n + 1)$ and consider the restriction $f : I_n \longrightarrow X - \{a\}$.

If $X - \{a\} \subseteq I_n$ then $X - \{a\} = I_n$, $a = n + 1$ and $X = I_{n+1}$.

Suppose $X - \{a\} \nsubseteq I_n$, then $n + 1 \in X - \{a\}$ and one can find b such that $f(b) = n + 1$. Let $g : I_{n+1} \longrightarrow X$ be the defined by $g(m) = f(m)$ if $m \neq n + 1, a$; $g(n+1) = n+1$; $g(b) = a$. By construction, the restriction $g : I_n \longrightarrow X - \{n+1\}$ is a bijection and obviously $X - \{n + 1\} \subseteq I_n$, hence $X - \{n + 1\} = I_n$ and it follows that $X = I_{n+1}$. □

Corollary 1.45. *(Number of elements is well-defined) If there is a bijection $f : I_n \longrightarrow I_m$ then $m = n$. Therefore, if $f : I_n \longrightarrow X$ and $g : I_m \longrightarrow X$ are bijections then $n = m$.*

Proof. The first part follows directly from the theorem. For the second part, consider the composition $(f^{-1} \circ g) : I_m \longrightarrow I_n$. □

Corollary 1.46. *There is no bijection $f : X \longrightarrow Y$ between a finite set X and a proper subset $Y \subseteq X$.*

Proof. By definition there is a bijection $\varphi : I_n \longrightarrow X$ for some $n \in \mathbb{N}$. Since Y is proper, $A := \varphi^{-1}(Y)$ is also proper in I_n. Let $\varphi_A : A \longrightarrow Y$ be the restriction of φ from I_n to A. Suppose there is a bijection $f : X \to Y$, then the composite function $\varphi_A^{-1} \circ f \circ \varphi : I_n \to A$ defines a bijection, a contradiction. □

Theorem 1.47. *Let X be a finite set and $Y \subseteq X$, then Y is finite and $|Y| \leq |X|$, the equality occurs only if $X = Y$.*

Proof. It's enough to prove the result for $X = I_n$. If $n = 1$ the result is obvious. Suppose the result is valid for I_n and consider $Y \subseteq I_{n+1}$. If $Y \subseteq I_n$, the induction hypothesis gives the result, so assume $n + 1 \in Y$. Then $Y - \{n + 1\} \subseteq I_n$ and by induction, there is a bijection $f : I_p \to Y - \{n + 1\}$, where $p \leq n$. Let $g : I_{p+1} \to Y$ be a bijection defined by $g(n) = f(n)$ if $n \in I_p$, and $g(p+1) = n+1$. This proves that Y is finite, moreover since $p \leq n \Rightarrow p+1 \leq n+1$. It follows by induction that $|Y| \leq n$. The last statement says that if $Y \subseteq I_n$ and $|Y| = n$ then $Y = I_n$, but this is a direct consequence of Theorem 1.44. □

The following corollaries are immediate:

Corollary 1.48. *Let Y be a finite set, and let $f : X \to Y$ be an injective function. Then X is also finite, and $|X| \leq |Y|$.*

Proof. Indeed, since $f(X) \subseteq Y$ and Y is finite, it follows that $f(X)$, and hence X, is finite and satisfies $|X| \leq |Y|$. □

Corollary 1.49. *Let X be a finite set, and let $f : X \to Y$ be a surjective function. Then Y is also finite, and $|Y| \leq |X|$.*

Proof. Since f is surjective, by the proof of Theorem 1.30, f has an injective right-inverse $g : Y \to X$. The result follows by the corollary above. □

A set X that is not finite is said to be **infinite**. More precisely, X is infinite if it is nonempty and there exists no bijection $f : I_n \to X$ for any $n \in \mathbb{N}$.

Example 1.50. The set of natural numbers $\mathbb{N}$ is infinite since there is no surjection from I_n onto $\mathbb{N}$. Indeed, for any function $f : I_n \to \mathbb{N}$, the number

$$f(1) + f(2) + \cdots + f(n) + 1$$

is not in the range of f.

Example 1.51. $\mathbb{Z}$ and $\mathbb{Q}$ are also infinite sets since they contain $\mathbb{N}$, which is infinite.

A set $X \subseteq \mathbb{N}$ is **bounded**, if there is a number $M \in \mathbb{N}$ such that $n \leq M$ for all $n \in X$.

Theorem 1.52. *Let $X \subseteq \mathbb{N}$ be nonempty. The following are equivalent:*

(a) X is finite;
(b) X is bounded;
(c) X has a greatest element.

Proof. The proof is based on the implications a ⇒ b, b ⇒ c, c ⇒ a.

(a $\Rightarrow$ b) Let $X = \{x_1, x_2, \ldots, x_n\}$. Then

$$x \leq M = \max\{x_1, x_2, \ldots, x_n\}$$

for every $x \in X$. Hence, X is bounded by M.

(b $\Rightarrow$ c) Consider the set

$$A = \{n \in \mathbb{N} \mid n \geq x \text{ for all } x \in X\}.$$

Since X is bounded, $A \neq \emptyset$. By the well-ordering principle, A has a minimum element, say $m \in A$.
If $m \in X$, then m is the greatest element of X.
Suppose instead that $m \notin X$. Then $m > x$ for all $x \in X$. Since $X \neq \emptyset$, $m > 1$, so we can write $m = p + 1$ for some $p \in \mathbb{N}$.
If $p \geq x$ for all $x \in X$, then $p \in A$, contradicting the minimality of m since $p < m$.
Otherwise, there exists $x \in X$ such that $x > p$. But then $x \geq m$, which contradicts $m \notin X$ unless $x = m$, which is impossible by assumption.
Therefore, it follows that $m \in X$, and m is the greatest element.

(c $\Rightarrow$ a) If X has a greatest element, say M, then $X \subseteq I_M$, which implies X is finite.

□

Theorem 1.53. *Let X and Y be two sets such that $|X| = m$, $|Y| = n$, and $X \cap Y = \emptyset$. Then $X \cup Y$ is finite and*

$$|X \cup Y| = m + n.$$

Proof. Since $|X| = m$, there exists a bijection

$$f : I_m \to X.$$

Similarly, since $|Y| = n$, there exists a bijection

$$g : I_n \to Y.$$

Because $X \cap Y = \emptyset$, the sets X and Y are disjoint. Define

$$h : I_{m+n} \to X \cup Y$$

by

$$h(k) = \begin{cases} f(k), & 1 \leq k \leq m, \\ g(k - m), & m + 1 \leq k \leq m + n. \end{cases}$$

We claim that h is a bijection.
Suppose $h(k_1) = h(k_2)$.
- If $k_1, k_2 \leq m$, then $f(k_1) = f(k_2)$, so $k_1 = k_2$ since f is injective.
- If $k_1, k_2 > m$, then $g(k_1 - m) = g(k_2 - m)$, so $k_1 = k_2$ since g is injective.

- If one of k_1, k_2 is $\leq m$ and the other $> m$, then $h(k_1) \in X$ and $h(k_2) \in Y$, contradicting $X \cap Y = \emptyset$.

It follows that h is injective.

For any $x \in X \cup Y$, either $x \in X$ or $x \in Y$. If $x \in X$, then $x = f(k) = h(k)$ for some $k \leq m$. If $x \in Y$, then $x = g(l) = h(m + l)$ for some $l \leq n$. Thus, h is surjective.

Therefore, h is a bijection and

$$|X \cup Y| = m + n.$$

□

The corollaries below follow immediately from the preceding theorem.

Corollary 1.54. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite and $X_i \cap X_j = \emptyset$ if $i \neq j$. Then $\bigcup_{i=1}^{n} X_i$ is finite and*

$$\left|\bigcup_{i=1}^{n} X_i\right| = \sum_{i=1}^{n} |X_i|.$$

Corollary 1.55. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite. Then $\bigcup_{i=1}^{n} X_i$ is finite and*

$$\left|\bigcup_{i=1}^{n} X_i\right| \leq \sum_{i=1}^{n} |X_i|$$

.

Proof. For each $i = 1, \ldots, n$, set $Y_i = X_i \times \{i\}$. Then the projection

$$\pi_i : \bigcup_{i=1}^{n} Y_i \to \bigcup_{i=1}^{n} X_i$$

in the first coordinate is surjective, by Corollaries 1.49 and 1.54, the proof is complete. □

Corollary 1.56. *Let $X_1, X_2, \ldots, X_n$, be a finite collection of sets such that each X_i is finite. Then $X_1 \times \ldots \times X_n$ is finite and*

$$|X_1 \times \ldots \times X_n| = \prod_{i=1}^{n} |X_i|.$$

Proof. It's suffices to prove for $n = 2$, since the general case follows from this one. Let $X_2 = \{y_1, \ldots, y_m\}$, notice that $X_1 \times X_2 = X_1 \times \{y_1\} \cup \ldots \cup X_2 \times \{y_m\}$, the result follows by Corollary 1.54. □

1.6 Countable and Uncountable Sets

A set X is **countable** if it is either finite or there exists a bijection $f : \mathbb{N} \to X$. In the latter case, since $\mathbb{N}$ is infinite, X is infinite, and we say that X is **countably infinite**.

Example 1.57. The set $X = \{ 2n \in \mathbb{N} \mid n \in \mathbb{N} \}$ of all even numbers is countable. The function $f(x) = 2x$ defines a bijection between X and $\mathbb{N}$.

Theorem 1.58. *Let X be an infinite set. Then X has a countably infinite subset.*

Proof. It suffices to construct an injective function $f : \mathbb{N} \to X$, since any injective function is a bijection onto its image.

Choose an element $a_1 \in X$ and define $f(1) = a_1$. Let $X_1 = X \setminus \{a_1\}$. Since X is infinite, X_1 is also infinite. Choose an element $a_2 \in X_1$ and set $f(2) = a_2$. Proceeding inductively, define $a_n \in X_{n-1}$, where

$$X_{n-1} = X \setminus \{a_1, a_2, \ldots, a_{n-1}\},$$

and set $f(n) = a_n$ for each $n \in \mathbb{N}$.

To show that f is injective, suppose $f(n) = f(m)$. Then $a_n = a_m$, which implies $n = m$, since the elements $a_1, a_2, \ldots$ were chosen to be distinct. Thus, f is injective.

□

Corollary 1.59. *A set X is infinite if and only if there is a bijection $f : X \to Y$, where $Y \subsetneq X$ is a proper subset.*

Proof. Suppose X infinite, by Theorem 1.58, X has a countably infinite subset, say $Z = \{a_1, a_2, a_3, \ldots\}$. Set $Y = (X - Z) \cup \{a_2, a_4, a_6, \ldots\}$ and define $f(x) = x$ if $x \in X - Z$, and $f(a_n) = a_{2n}$ otherwise. The function $f(x)$, defined this way, is clearly a bijection. The converse follows from Corollary 1.46. □

A function $f : X \to Y$ is called *increasing* if $x < y \Rightarrow f(x) < f(y)$.

Theorem 1.60. *Every subset X of $\mathbb{N}$ is countable.*

Proof. The proof is very similar to the one in Theorem 1.58. If X is finite then is countable, so assume X infinite. We define an increasing bijection $f : \mathbb{N} \to X$ by induction. Let $X_1 = X$, $a_1 = \min X$ (which exists by Theorem 1.39), and set $f(1) = a_1$. Now, define $X_2 = X - \{a_1\}$ and $f(2) = a_2 = \min X_2$. By induction, we define $f(n) = a_n = \min X_n$, where $X_n = X - \{a_1, a_2, \ldots, a_{n-1}\}$. The function $f(n)$ is injective by construction, suppose $f(n)$ not surjective. There is $x \in X$ such that $x \notin f(\mathbb{N})$. So $x \in X_n$ for every n, which implies that $x > f(n)$ for every n, and x is a bound for the infinite set $f(\mathbb{N})$, a contradiction by Theorem 1.52. □

Example 1.61. Let X be a countable set. Then by Theorem 1.60, for any $Y \subseteq X$, Y is countable.

Example 1.62. The set of all prime numbers is countable, as it is a subset of $\mathbb{N}$.

Example 1.63. Let Y be a countable set and let $f : X \to Y$ be an injective function. Then X is countable. Indeed, since f is injective, it defines a bijection between X and its image $f(X) \subseteq Y$. As every subset of a countable set is countable, it follows that $f(X)$, and hence X, is countable. Similarly, if X be a countable set and $f : X \to Y$ a surjective function. Then Y is countable.

Example 1.64. The set $\mathbb{Z}$ of integers is countable. Indeed, consider the function $f : \mathbb{Z} \to \mathbb{N}$ defined by

$$f(m) = \begin{cases} 1, & \text{if } m = 0, \\ 2m, & \text{if } m > 0, \\ -2m + 1, & \text{if } m < 0. \end{cases}$$

This function is bijective, and hence $\mathbb{Z}$ is countable.

Example 1.65. The set $\mathbb{N} \times \mathbb{N}$ is countable. Indeed, the function

$$f(m, n) = 2^m 3^n$$

defines an injective mapping $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$. Since the image of f is a subset of $\mathbb{N}$, which is countable, it follows that $\mathbb{N} \times \mathbb{N}$ is countable.

Corollary 1.66. *Let $X_1, X_2, \ldots$ be a countable collection of countable sets. Set $X = \bigcup_{i=1}^{\infty} X_i$, then X is countable.*

Proof. Let $f_i : \mathbb{N} \to X_i$ be a counting function for each $i \in \mathbb{N}$. Then $f(i, m) := f_i(m)$ defines a surjection $f : \mathbb{N} \times \mathbb{N} \to X$. By Example 1.63, X is countable. □

Corollary 1.67. *If X, Y are countable sets then $X \times Y$ is countable.*

Proof. Let $f_1 : \mathbb{N} \to X$, $f_2 : \mathbb{N} \to Y$ be counting functions. Then $f(m, n) := (f_1(m), f_2(n))$ defines a bijection, Example 1.65 concludes the proof. □

Corollary 1.68. *The set $\mathbb{Q}$ of rational numbers is countable.*

Proof. Let $\mathbb{Z}^*$ denote the set of nonzero integers. Define the surjective function $f : \mathbb{Z} \times \mathbb{Z}^* \to \mathbb{Q}$ given by $f(m, n) = \frac{m}{n}$. By Example 1.63, $\mathbb{Q}$ is countable. □

A set X is **uncountable** if it is not countable. Given two sets X and Y, if there exists a bijection $f : X \to Y$, we say that X and Y have the same **cardinality**, and write:

$$\text{card}(X) = \text{card}(Y).$$

If there exists an injective function $f : X \to Y$ but no surjective function $g : X \to Y$, then we write

$$\text{card}(X) < \text{card}(Y).$$

The cardinality of the set of natural numbers $\mathbb{N}$ is denoted by

$$\mathrm{card}(\mathbb{N}) = \aleph_0.$$

If a set X is finite with exactly n elements, we write $\mathrm{card}(X) = n$. By definition, for any infinite set X, we have:

$$\aleph_0 \le \mathrm{card}(X).$$

Recall that given two sets X and Y, the set $\mathcal{F}(X,Y)$ denotes the set of all functions from X to Y.

Theorem 1.69. *(Cantor) Let X and Y be sets, with Y containing at least two elements. Then there is no surjective function $\varphi : X \to \mathcal{F}(X,Y)$.*

Proof. Suppose a function $\phi : X \to \mathcal{F}(X,Y)$ is given and let $\phi_x = \phi(x) : X \to Y$ be the image of $x \in X$, which itself is a function. We claim that there is a $f : X \to Y$ that is not ϕ_x for any X. Indeed, for each $x \in X$ let $f(x)$ be an element different than $\phi_x(x)$ (this is possible sice $|Y| \ge 2$), then $f \ne \phi_x$ for every $x \in X$ and hence, ϕ is not surjective. □

Corollary 1.70. *Given any set X, we have*

$$\mathrm{card}(X) < \mathrm{card}(\mathcal{P}(X)).$$

In particular,

$$\aleph_0 < \mathrm{card}(\mathcal{P}(\mathbb{N})).$$

Proof. Let $Y = \{0,1\}$. Then $\mathcal{F}(X,Y)$ is in bijection with $\mathcal{P}(X)$, since each function $f : X \to \{0,1\}$ corresponds to the subset $A = f^{-1}(1) \subseteq X$. By Cantor's Theorem (Theorem 1.69), there is no surjective function $\varphi : X \to \mathcal{F}(X,Y)$. It follows that

$$\mathrm{card}(X) < \mathrm{card}(\mathcal{F}(X,Y)) = \mathrm{card}(\mathcal{P}(X)).$$

The inequality $\aleph_0 < \mathrm{card}(\mathcal{P}(\mathbb{N}))$ is a particular case. □

Corollary 1.71. *Let $X_1, X_2, \ldots$ be a countable collection of countably infinite sets. Then the infinite cartesian product $X = \prod_{i=1}^{\infty} X_i$ is uncountable.*

Proof. It's enough to prove the result for $X_i = \mathbb{N}$. In this case, $X = \mathcal{F}(\mathbb{N},\mathbb{N})$ and the result follows from Theorem 1.69. □

Example 1.72. The set $X = \mathcal{F}(\mathbb{N},\mathbb{N}) = \{a_n\,;\, a_n \in \mathbb{N}\}$ is the set of all sequences of natural numbers. By the corollary above, this set is uncountable. Similarly, the power set of the natural numbers $\mathcal{P}(\mathbb{N})$ is uncountable.

Example 1.73. Any non degenerate interval (a,b) is uncountable. More generally, the set of all real numbers $\mathbb{R}$ is uncountable. This will be proved in the next sections.

Exercises

1. Let A, B, X be sets with the following properties:

   $A \subseteq X$ and $B \subseteq X$
   For any set Y if $A \subseteq Y$ and $B \subseteq Y$ then $X \subseteq Y$.

   Show that $X = A \cup B$.
2. Given $A, B \subseteq E$, show that $A \subseteq B$ if and only if $A \cap B^c = \emptyset$.
3. Give examples of sets A, B, C such that $(A \cup B) \cap C \neq A \cup (B \cap C)$.
4. Show that $A = B$ if and only if $(A \cap B^c) \cup (A^c \cap B) = \emptyset$.
5. Given two sets A, B we define the *symmetric difference* $A \Delta B$ by

$$A \Delta B = (A - B) \cup (B - A).$$

   Prove that if $A \Delta B = A \Delta C$, then $B = C$.
6. Show that $(A \cup B) \times C = (A \times C) \cup (B \times C)$.
7. Show that a function $f : A \to B$ is injective if and only if $f(A - X) = f(A) - f(X)$ for every $X \subseteq A$.
8. Let $f : A \to B$ be given. Show that

   a. For every $Z \subseteq B$, we have $f(f^{-1}(Z)) \subseteq Z$.
   b. $f(x)$ is surjective if and only if $f(f^{-1}(Z)) = Z$ for every $Z \subseteq B$.

9. Given a family of sets $(A_\lambda)_{\lambda \in L}$, let X be a set with the following properties:

   1. For every $\lambda \in L$, we have $A_\lambda \subseteq X$.
   2. If $A_\lambda \subseteq Y$ for every $\lambda \in L$, then $X \subseteq Y$.

   Show that $X = \bigcup_{\lambda \in L} A_\lambda$.
10. Let $f : \mathcal{P}(A) \to \mathcal{P}(A)$ be a function such that if $X \subseteq Y$ then $f(Y) \subseteq f(X)$ and $f(f(X)) = X$. Show that $f(\bigcup_{\lambda \in L} X_\lambda) = \bigcap f(X_\lambda)$ and $f(\bigcap_{\lambda \in L} X_\lambda) = \bigcup f(X_\lambda)$. [Here X, Y, X_λ are subsets of A]
11. Let $\mathcal{F}(X; Y)$ denote the set of all functions with domain X and codomain Y. Given the sets A, B, C, show that there is a bijection

$$\mathcal{F}(A \times B; C) \to \mathcal{F}(A; \mathcal{F}(B; C)).$$

12. Given two natural numbers $a, b \in \mathbb{N}$, prove that there is a natural number $m \in \mathbb{N}$ such that $m \cdot a > b$.
13. Let $a \in \mathbb{N}$. If the set X has the following property: $a \in X$ and $n \in X \Rightarrow n + 1 \in X$. Then X contains all natural numbers greater than or equal to a.
14. A number $a \in \mathbb{N}$ is called **predecessor** of $b \in \mathbb{N}$ if $a < b$ and there is no $c \in \mathbb{N}$ such that $a < c < b$. Prove that every number, except 1, has a predecessor.
15. Show the following using induction:

   a. $2(1 + 2 + \ldots + n) = n(n + 1)$;

b. $1 + 3 + 5 + \ldots + (2n + 1) = (n + 1)^2$;
c. $n \geq 4 \Rightarrow n! > 2^n$.

16. Using strong induction show that the decomposition of any number in prime factors is unique.
17. Let X be a finite set with n elements. Use induction to show that the set of all functions $f : X \longrightarrow X$ has exactly $n!$ elements.
18. Let X be a finite set. Show that a function $f : X \longrightarrow X$ is injective $\iff$ is surjective.
19. Give an example of a surjective function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $f^{-1}(n)$ is infinite.
20. Show that the power set $\mathcal{P}(A)$ of a set A with n elements has 2^n elements.
21. Show that if A is countably infinite then $\mathcal{P}(A)$ is uncountable.
22. Let $f : X \longrightarrow X$ be injective but not surjective. If $x \in X - f(X)$, show that $x, f(x), f(f(x)), \ldots$ are pairwise distinct.
23. Let X be an infinite set e Y a finite set. Show that there is a surjective function $f : X \longrightarrow Y$ and an injective function $g : Y \longrightarrow X$.
24. Find subsets $X_i \subseteq \mathbb{N}$ and a decomposition

$$\mathbb{N} = X_1 \cup X_2 \cup \ldots \cup X_i \cup \ldots,$$

such that X_i are infinite sets and pairwise disjoints.
25. Let $X \subseteq \mathbb{N}$ be infinite. Show that there is a unique increasing bijection $f : \mathbb{N} \rightarrow X$.
26. A sequence of natural numbers $\{a_1, a_2, a_3, \ldots\}$ is called increasing if $a_i < a_{i+1}$. Show that the set of all increasing sequences of natural numbers is uncountable.
27. (Cantor-Bernstein-Schroder theorem) Given sets A and B, let $f : A \longrightarrow B$ and $g : B \longrightarrow A$ be injective functions. Show that there is a bijection $h : A \longrightarrow B$.
28. Given a sequence of sets $A_1, A_2, A_3, \ldots$, we define the *limit superior* as the set

$$\limsup A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} A_i \right).$$

Similarly, the *limit inferior* is the set

$$\liminf A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} A_i \right).$$

a. Show that $\limsup A_n$ is the set of elements that belong to A_i for infinitely many values of i. Similarly, show that $\liminf A_n$ is the set of elements that belong to A_i for every value of i, except possibly, for a finite number of values of i.
b. Conclude that $\liminf A_n \subseteq \limsup A_n$.

c. Show that if $A_n \subseteq A_{n+1}$ for every n then $\liminf A_n = \limsup A_n = \bigcup_{n=1}^{\infty} A_n$.
d. Show that if $A_{n+1} \subseteq A_n$ for every n then $\liminf A_n = \limsup A_n = \bigcap_{n=1}^{\infty} A_n$.
e. Give an example of sequence such that $\liminf A_n \neq \limsup A_n$.

Chapter 2
The real numbers $\mathbb{R}$

This chapter introduces one of the most fundamental algebraic structures in analysis: the field of real numbers. We begin by presenting the definition of a field through a precise set of axioms governing addition and multiplication. These axioms encapsulate, in a formal way, the essential algebraic properties that we are familiar with.

We then introduce ordered fields, which not only support algebraic operations but also possess a notion of order. This ordering allows for a meaningful comparison between elements and enables the study of limits, inequalities, and convergence.

The chapter continues by exploring intervals, absolute value, and foundational ideas like bounds, supremum, and infimum within ordered fields. This leads naturally to the central concept of completeness, which distinguishes the real numbers $\mathbb{R}$ from all other ordered fields.

The final sections delve into the density and uncountability of the real numbers, showing that between any two real numbers there lie infinitely many rationals—and infinitely many irrationals. These results not only underscore the richness of the real line but also hint at the profound structure underlying the continuum.

2.1 Fields

A **field** K is a set K together with two operations, called addition and multiplication,

$$\begin{aligned} + : & K \times K \to K \\ & (x, y) \mapsto x + y \end{aligned} \quad \text{and} \quad \begin{aligned} \cdot : & K \times K \to K \\ & (x, y) \mapsto x \cdot y \end{aligned}$$

satisfying the following properties (also called *field axioms*):

Axioms of addition

For all $x, y, z \in K$:

- ❑ $x + (y + z) = (x + y) + z$ *(associativity)*;
- ❑ $x + y = y + x$ *(commutativity)*;
- ❑ There exists an element $0 \in K$ such that $x + 0 = x$ *(identity element)*;
- ❑ For each $x \in K$, there exists $y \in K$ such that $x + y = 0$. We define $-x := y$, and write $z - x$ in place of $z + (-x)$ *(inverse element)*;

Axioms of Multiplication

For all $x, y, z \in K$:

- ❑ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *(associativity)*;
- ❑ $x \cdot y = y \cdot x$ *(commutativity)*;
- ❑ There exists an element $1 \in K$, with $1 \neq 0$, such that $x \cdot 1 = x$ *(identity element)*;
- ❑ For each $x \in K$ with $x \neq 0$, there exists $y \in K$ such that $x \cdot y = 1$. We define $x^{-1} := y$, and write $\frac{z}{x}$ in place of $z \cdot x^{-1}$ *(inverse element)*;
- ❑ $x \cdot (y + z) = x \cdot y + x \cdot z$ *(distributivity)*.

Example 2.1. The set of rational numbers $\mathbb{Q}$, equipped with the operations

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd},$$

forms a field.

In this field:

- The additive identity is $0 := \frac{0}{1}$;
- The multiplicative identity is $1 := \frac{1}{1}$;
- The multiplicative inverse of a nonzero element $\frac{a}{b}$ is given by

$$\left(\frac{a}{b}\right)^{-1} := \frac{b}{a}, \quad \text{for } a \neq 0.$$

Example 2.2. Let p be a prime number. The set of integers modulo p, denoted

$$\mathbb{Z}_p := \{\bar{0}, \bar{1}, \ldots, \overline{p-1}\},$$

equipped with the operations

$$\bar{a} + \bar{b} := \overline{a + b}, \quad \text{and} \quad \bar{a} \cdot \bar{b} := \overline{a \cdot b},$$

forms a field.

In this field:

- The additive identity is $0 := \bar{0}$;
- The multiplicative identity is $1 := \bar{1}$.

Moreover, for any $\bar{a} \in \mathbb{Z}_p$ with $\bar{a} \neq \bar{0}$, Fermat's Little Theorem implies

$$\bar{a}^{p-1} = \bar{1},$$

so multiplying both sides by $\bar{a}^{-1}$ yields

$$\bar{a}^{p-2} = \bar{a}^{-1}.$$

Thus, the multiplicative inverse of any nonzero element $\bar{a} \in \mathbb{Z}_p$ is given by

$$\bar{a}^{-1} = \bar{a}^{p-2}.$$

Example 2.3. The set of rational functions

$$\mathbb{Q}(t) := \left\{ \frac{p(t)}{q(t)} \;\middle|\; p(t), q(t) \in \mathbb{Q}[t],\ q(t) \not\equiv 0 \right\},$$

where $\mathbb{Q}[t]$ denotes the ring of polynomials with rational coefficients, forms a field under the usual operations:

$$\frac{p(t)}{q(t)} + \frac{r(t)}{s(t)} := \frac{p(t)s(t) + q(t)r(t)}{q(t)s(t)}, \quad \frac{p(t)}{q(t)} \cdot \frac{r(t)}{s(t)} := \frac{p(t)r(t)}{q(t)s(t)}.$$

Theorem 2.4. *(Properties of fields) Let K be a field and let $x, y, z \in K$. Then:*

- ❑ $x \cdot 0 = 0$;
- ❑ *If* $x \cdot z = y \cdot z$ *and* $z \neq 0$, *then* $x = y$;
- ❑ *If* $x \cdot y = 0$, *then* $x = 0$ *or* $y = 0$;
- ❑ *If* $x^2 = y^2$, *then* $x = y$ *or* $x = -y$.

Proof. Given $x \in K$,

$$x \cdot 0 + x = x \cdot (0 + 1) = x,$$

thus, $x \cdot 0 = 0$.

Similarly, if $z \neq 0$:

$$x = x \cdot z \cdot z^{-1} = y \cdot z \cdot z^{-1} = y.$$

Next, suppose $x \cdot y = 0$ but $x \neq 0$, then

$$x \cdot 0 = 0 = x \cdot y,$$

the item above implies $y = 0$. By symmetry, the result also holds when $y \neq 0$. Lastly,

$$x^2 = y^2 \Rightarrow x^2 - y^2 = 0 \Rightarrow (x - y)(x + y) = 0,$$

and it follows that $x = y$ or $x = -y$. □

2.2 Ordered Fields

An *ordered field* is a field K together with a subset $P \subseteq K$, called the set of *positive elements*, such that the following properties hold for all $x, y \in P$:

- ❑ $x + y \in P$ and $x \cdot y \in P$;
- ❑ For every $x \in K$, exactly one of the following holds:

$$x = 0, \quad x \in P, \quad \text{or} \quad -x \in P.$$

If we denote $-P := \{-p \mid p \in P\}$, then the field K can be written as the disjoint union

$$K = P \cup (-P) \cup \{0\}.$$

Notice that in an ordered field, if $x \neq 0$, then $x^2 \in P$. In particular, this implies $1 \in P$.

Example 2.5. The field of rational numbers $\mathbb{Q}$ together with the set

$$P = \left\{ \frac{a}{b} \in \mathbb{Q}\,;\ a \cdot b \in \mathbb{N} \right\}$$

is an ordered field.

Example 2.6. The field $\mathbb{Z}_p$ can't be ordered, since if we add $\bar{1}$, p times, the result is $\bar{0}$, i.e. $\bar{1} + \cdot + \bar{1} = \bar{0}$, but in an ordered field the sum of positive elements has to be positive, in particular nonzero.

Example 2.7. The field $\mathbb{Q}(t)$ of Example 2.3 together with the set

$$P = \left\{ \frac{p(t)}{q(t)}\,;\ \text{the leading coefficient of } p(t) \cdot q(t) \text{ is positive} \right\}$$

is an ordered field.

In an ordered field K, if $x - y \in P$, we write $x > y$ (or equivalently, $y < x$). In particular, $x > 0$ implies $x \in P$, and $x < 0$ implies $x \in -P$.
Notice that if $x \in P$ and $y \in -P$, then $x > y$.
The notation $x \leq y$ is used to indicate that $x < y$ or $x = y$; similarly, $x \geq y$ means $x > y$ or $x = y$.

Theorem 2.8. *Let K be an ordered field and let $x, y, z \in K$. Then:*

- ❑ *If $x < y$ and $y < z$, then $x < z$;*
- ❑ *If $x < y$, then $x + z < y + z$;*
- ❑ *Exactly one of the following holds: $x = y$, $x < y$, or $x > y$;*
- ❑ *If $z > 0$, then $x < y \Rightarrow x \cdot z < y \cdot z$; if $z < 0$, then $x < y \Rightarrow x \cdot z > y \cdot z$.*

Proof. The first two properties follow immediately from the definition of an ordered field. We now prove the last two.

Assume $x \neq y$. Then $x - y \neq 0$, and by trichotomy, either $x - y \in P$ or $y - x \in P$. Thus, exactly one of the relations $x = y$, $x < y$, or $x > y$ holds.

Now suppose $z > 0$ and $x < y$, so that $y - x \in P$. Then

$$y \cdot z - x \cdot z = (y - x) \cdot z \in P,$$

since P is closed under multiplication. Hence, $x \cdot z < y \cdot z$.

If $z < 0$, the result follows similarly, using the fact that $-z \in P$. □

Given two fields K and L, a function $f : K \to L$ is a *homomorphism*, if $\forall x, y \in K$, the following conditions hold:

$$f(x + y) = f(x) + f(y)$$
$$f(x \cdot y) = f(x) \cdot f(y)$$

We say f is an *isomorphism* if, in addition, f is bijective and f^{-1} is also a homomorphism. An *automorphism* $f : K \to K$ is an isomorphism between K and itself.

Example 2.9. For any field K, the isomorphism $i : K \to K$ given by $i(x) = x$ is an automorphism, often called the *trivial automorphism*.

Example 2.10. Let $K = \mathbb{Z}_p$ for some prime p. Then the function $x \mapsto x^p$ is an automorphism of $\mathbb{Z}_p$, called the *Frobenius automorphism*. The Frobenius automorphism plays an important role in commutative algebra and number theory, particularly in the study of fields of characteristic p and Galois theory.

Since in an ordered field K, the element 1 is always positive, we have $1 + 1 > 1 > 0$ and $1 + 1 + 1 > 1 + 1$. Thus, we can define an increasing injection

$$f : \mathbb{N} \to K$$

by $f(n) = \overbrace{1 + 1 + \cdots + 1}^{n \text{ times}}$, or more precisely, $f(1) = 1$ and $f(n+1) = f(n) + 1$. Therefore, it makes sense to identify $\mathbb{N}$ with $f(\mathbb{N}) \subseteq K$, and henceforth we will simply write

$$\mathbb{N} \subseteq K$$

whenever K is an ordered field.

Notice in particular that $f(n) \neq 0$ for any $n \in \mathbb{N}$, so every ordered field is infinite. More generally, whenever $f(n) \neq 0$ for all n, we say that K has **characteristic zero**. Hence, every ordered field has characteristic zero.

If we drop the assumption that the field K is ordered and there exists a number p such that $f(p) = 0$, then we say that K has **characteristic** p.

Example 2.11. The field $\mathbb{Q}$ clearly has characteristic zero. On the other hand, the field $\mathbb{Z}_p$ has characteristic p. More generally, if a field K has characteristic $p \neq 0$, then p must be a prime number.

Indeed, suppose p is the minimal positive integer such that

$$\underbrace{1+1+\cdots+1}_{p \text{ times}} = 0,$$

but p is not prime. Then we can write $p = ab$ for some integers $a, b > 1$. Consider

$$0 = \underbrace{1+1+\cdots+1}_{ab \text{ times}} = \left(\underbrace{1+1+\cdots+1}_{a \text{ times}}\right) \cdot \left(\underbrace{1+1+\cdots+1}_{b \text{ times}}\right).$$

Since K is a field and has no zero divisors, it follows that one of the two factors must be zero. Thus, either

$$\underbrace{1+1+\cdots+1}_{a \text{ times}} = 0 \quad \text{or} \quad \underbrace{1+1+\cdots+1}_{b \text{ times}} = 0,$$

contradicting the minimality of p. Therefore, p must be prime.

We may extend the injection described above to a function $f : \mathbb{Z} \to K$, allowing us to view $\mathbb{Z} \subseteq K$ as well. Hence, we have

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq K.$$

Finally, we can use the map $f : \mathbb{Z} \to K$ to define an injection $g : \mathbb{Q} \to K$ by

$$g\left(\frac{a}{b}\right) := f(a) \cdot f(b)^{-1},$$

where $a \in \mathbb{Z}$, $b \in \mathbb{N}$, and $b \neq 0$. In this way, we may identify $\mathbb{Q}$ with $g(\mathbb{Q}) \subseteq K$, and write

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq K$$

whenever K is an ordered field.

Notice that if $K = \mathbb{Q}$ in the above discussion, then $g : \mathbb{Q} \to \mathbb{Q}$ is the trivial automorphism. i.e.,

$$g\left(\frac{a}{b}\right) = \frac{a}{b}.$$

Theorem 2.12. *(Bernoulli's inequality) Let K be an ordered field and $x \in K$. If $x \geq -1$ and $n \in \mathbb{N}$, then*

$$(1+x)^n \geq 1 + n \cdot x$$

Proof. We use induction on $n \in \mathbb{N}$. The case $n = 1$ is clear, suppose the result valid for n. Then $(1+x)^{n+1} = (1+x)^n(1+x) \geq (1+n\cdot x)(1+x) = 1+x+n\cdot x+x^2 \geq 1 + x + n \cdot x$, as expected. *(Notice that we used the fact that $x \geq -1$ in the first inequality and Theorem 2.8.)* □

Let K be an ordered field and $a < b$ be elements of K. Any subset of the following form is called an *interval*:

- ❑ $[a, b] = \{x \in K \; ; \; a \leq x \leq b\}$ *(closed interval)*
- ❑ $(a, b) = \{x \in K \; ; \; a < x < b\}$ *(open interval)*
- ❑ $[a, b) = \{x \in K \; ; \; a \leq x < b\}$ and $(a, b] = \{x \in K \; ; \; a < x \leq b\}$
- ❑ $(-\infty, b) = \{x \in K \; ; \; x < b\}$ and $(-\infty, b] = \{x \in K \; ; \; x \leq b\}$
- ❑ $(a, \infty) = \{x \in K \; ; \; a < x\}$ and $[a, \infty) = \{x \in K \; ; \; a \leq x\}$
- ❑ $(-\infty, \infty) = K$

When $a = b$, we define $[a, a] = \{a\}$ *(degenerate interval)* and $(a, a) = \emptyset$.

Let K be an ordered field and $x \in K$. The *absolute value* of x, denoted by $|x|$, is defined by

$$|x| := \max\{x, -x\},$$

that is, $|x|$ is the greater of the two elements x and $-x$.

The following Theorem is an immediate consequence of the preceding definitions. The proof is left as an exercise to the reader.

Theorem 2.13. *Let x, y be elements of an ordered field K. The following statements are equivalent:*

(1) $-y \leq x \leq y$
(2) $x \leq y$ *and* $-x \leq y$
(3) $|x| \leq y$

Corollary 2.14. *Let $x, a, \varepsilon \in K$, where K is an ordered field. Then*

$$|x - a| \leq \varepsilon \Leftrightarrow a - \varepsilon \leq x \leq a + \varepsilon.$$

Proof. Suppose $|x - a| \leq \varepsilon$. By definition of absolute value, this means

$$-\varepsilon \leq x - a \leq \varepsilon.$$

Adding a to each part of the inequality yields

$$a - \varepsilon \leq x \leq a + \varepsilon.$$

Conversely, suppose $a - \varepsilon \leq x \leq a + \varepsilon$. Subtracting a throughout gives

$$-\varepsilon \leq x - a \leq \varepsilon,$$

which implies $|x - a| \leq \varepsilon$, as desired. □

Remark.

The Theorem and corollary remains valid if we exchange $\leq$ by $<$.

Theorem 2.15. *Let x, y, z be elements of an ordered field K. Then*

(1) $|x + y| \leq |x| + |y|$;
(2) $|x \cdot y| = |x| \cdot |y|$;
(3) $|x| - |y| \leq ||x| - |y|| \leq |x - y|$;
(4) $|x - z| \leq |x - y| + |y - z|$.

Proof. (1) If x and y have the same sign or one of them is zero then we obviously have $|x+y| = |x| + |y|$. Otherwise, suppose they have opposite sign and $|x| > |y|$. Then $|x+y| = |x| - |y| \leq |x| + |y|$. If instead $|x| < |y|$, we may reverse the roles of x and y and apply the same argument to obtain the same inequality.

(2) The result is clear if x and y have the same sign or one of them is zero. Suppose they have opposite sign, say $x > 0$ and $y < 0$. Then

$$|x \cdot y| = -x \cdot y = x \cdot -y = |x| \cdot |y|$$

(3) The first inequality is clear, we prove the second one. Apply (1) with $x - y$ and y to obtain

$$|x| \leq |x-y| + |y| \Rightarrow |x| - |y| \leq |x-y|.$$

Similarly, $|y| - |x| \leq |x-y|$ and the conclusion follows.

(4) Apply (1) with $x - z$ and $z - y$ instead of x and y.

□

Let K be an ordered field and let $X \subseteq K$. An element $M \in K$ is called an **upper bound** of X if $x \leq M$ for every $x \in X$. Similarly, an element $m \in K$ is called a **lower bound** of X if $m \leq x$ for every $x \in X$.

We say that X is *bounded from above* if it has an upper bound, *bounded from below* if it has a lower bound, and simply *bounded* if it has both an upper and a lower bound, that is, if $X \subseteq [m, M]$ for some $m, M \in K$.

Example 2.16. The well-ordering principle guarantees that $\mathbb{N}$ is bounded from below when viewed as a subset of the ordered field $\mathbb{Q}$. On the other hand, $\mathbb{N}$ is clearly not bounded from above in $\mathbb{Q}$, since for any $n \in \mathbb{N}$, we have $n + 1 > n$.

Example 2.17. Oddly enough, the set $\mathbb{N}$ is bounded from above in the ordered field $\mathbb{Q}(t)$ introduced in Example 2.7. Indeed, for any $n \in \mathbb{N}$, the rational function $r(t) = t$ satisfies $r(t) - n > 0$. Hence, $r(t) \in \mathbb{Q}(t)$ serves as an upper bound for $\mathbb{N}$, implying that $\mathbb{N}$ is bounded from above, and consequently bounded, in $\mathbb{Q}(t)$.

Theorem 2.18. *Let K be an ordered field. The following statements are equivalent:*

1. *$\mathbb{N}$ is not bounded from above in K;*
2. *For any $a, b \in K$ with $a > 0$, there exists $n \in \mathbb{N}$ such that $n \cdot a > b$;*
3. *For any $a > 0$ in K, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$.*

A field K satisfying the above conditions is called ***Archimedean field****.*

Proof. The proof is based on the implications $1 \Rightarrow 2$, $2 \Rightarrow 3$, and $3 \Rightarrow 1$.

($1 \Rightarrow 2$) Since $\mathbb{N}$ is unbounded from above in K, for given $a, b \in K$ with $a > 0$, there exists $n \in \mathbb{N}$ such that

$$\frac{b}{a} < n,$$

hence multiplying both sides by a gives

$$n \cdot a > b.$$

(2 ⇒ 3) Taking $b = 1$ in statement 2, we obtain that for any $a > 0$, there exists $n \in \mathbb{N}$ such that

$$n \cdot a > 1,$$

which is equivalent to

$$\frac{1}{n} < a,$$

and since $n \in \mathbb{N}$, $1/n > 0$, so

$$0 < \frac{1}{n} < a.$$

(3 ⇒ 1) Given any $a > 0$, consider $\frac{1}{a}$. By statement 3, there exists $n \in \mathbb{N}$ such that

$$\frac{1}{n} < \frac{1}{a} \Leftrightarrow n > a.$$

This shows that for every $a \in K$, there is some natural number n greater than a, so $\mathbb{N}$ is not bounded from above.
Similarly, no negative element can be an upper bound of $\mathbb{N}$, because if $x < 0$, then $-x > 0$ and the same argument applies.

□

Remark.

Examples 2.16 and 2.17 say that $\mathbb{Q}$ is Archimedean but $\mathbb{Q}(t)$ isn't.

2.3 The real field $\mathbb{R}$

Let K be an ordered field and $X \subseteq K$ be a bounded from above subset. The **supremum** of X, denoted $\sup X$ is the least upper bound of X, in other words, among all upper bounds $M \in K$ of X, i.e. $x \leq M$ for every $x \in X$, $\sup X \in K$ is the least of them. Therefore, $\sup X \in K$ has the following properties:

(i) (upper bound) For every $x \in X$, $x \leq \sup X$.
(ii) (least upper bound) Given any $a \in K$ such that $x \leq a$ for every $x \in X$, then $\sup X \leq a$. In other words, if $a < \sup X$ then $\exists b \in X$ such that $a < b$.

Lemma 2.19. *If the supremum of a set X exists, then it is unique.*

Proof. Suppose $a = \sup X$ and $b = \sup X$. By property (ii) above, $a \leq b$ since a is the least upper bound of X. Similarly, since b is also the least upper bound, we have $b \leq a$. Therefore, $a = b$. □

Lemma 2.20. *If a set* X *has a maximum element, then* $\max X = \sup X$.

Proof. Indeed, $\max X$ is obviously an upper bound and any other upper bound is greater than or equal to the maximum. □

Example 2.21. Consider the set $I_n = \{1, 2, \ldots, n\} \subseteq \mathbb{Q}$. Then

$$\sup I_n = \max I_n = n.$$

Example 2.22. Consider the set

$$X = \left\{-\frac{1}{n}; n \in \mathbb{N}\right\} \subseteq \mathbb{Q}.$$

Then $\sup X = 0$. Indeed, 0 is an upper bound for X, and given any $a < 0$, we can find $n \in \mathbb{N}$ such that $-\frac{1}{n} > a$, since $\mathbb{Q}$ is an Archimedean field. Hence, no number less than 0 can be an upper bound of X, so 0 is the least upper bound.

Similar to the idea of supremum, the **infimum** of a bounded from below set $X \subseteq K$, denoted $\inf X$, is the greatest lower bound. The element $\inf X \in K$ has the following properties:

(i) (lower bound) For every $x \in X$, $x \geq \inf X$.
(ii) (greatest lower bound) Given any $a \in K$ such that $x \geq a$ for every $x \in X$, then $\inf X \geq a$.

The lemmas 2.19 and 2.20 extend naturally to the notion of infimum, namely, if $X \subseteq K$ has a minimum element m then $m = \inf X$. Additionally, the infimum is unique.

This discussion leads to the following Theorem:

Theorem 2.23. *Let* $X \subseteq K$ *be a bounded subset of an ordered field* K. *Then,*

$$\inf X \in X \Leftrightarrow \inf X = \min X$$

and

$$\sup X \in X \Leftrightarrow \sup X = \max X.$$

In particular, every finite set has a supremum and infimum.

Example 2.24. Consider the set $X = (a, b)$, an open interval in a ordered field K. Then $\inf X = a$ and $\sup X = b$. Indeed, a is a lower bound, by definition of interval, suppose $c > a$, we claim c can't be a lower bound. For instance, consider $d = \frac{a+c}{2} \in (a, b)$. We have $d < c$ if $c < b$, hence the conclusion.

Example 2.25. Let $X = \left\{\frac{1}{2^n} \mid n \in \mathbb{N}\right\} \subseteq \mathbb{Q}$. Then $\inf X = 0$ and $\sup X = \frac{1}{2}$. Notice that $\max X = \frac{1}{2}$, so by Lemma 2.20, $\sup X = \frac{1}{2}$.

Now, 0 is clearly a lower bound. Suppose $c > 0$. Since $\mathbb{Q}$ is Archimedean, we can find $n \in \mathbb{N}$ such that $n + 1 > \frac{1}{c}$. By Bernoulli's inequality (Theorem 2.12), we have

$$2^n = (1+1)^n \geq 1 + n > \frac{1}{c},$$

hence $c > \frac{1}{2^n}$, and so c cannot be a lower bound. It follows that $\inf X = 0$.

Lemma 2.26. *(Pythagoras) There is no $x \in \mathbb{Q}$ satisfying $x^2 = 2$.*

Proof. Suppose, for contradiction, that $x = \frac{p}{q} \in \mathbb{Q}$ satisfies $x^2 = 2$, where $p, q \in \mathbb{Z}$, $q \neq 0$, and the fraction is in lowest terms. Then

$$\left(\frac{p}{q}\right)^2 = 2 \quad \Rightarrow \quad p^2 = 2q^2.$$

Now, consider the prime factorizations of both sides. Since p^2 and q^2 are perfect squares, their factorizations contain an even number of each prime, including the prime 2. However, $2q^2$ introduces one additional factor of 2, making the total number of factors of 2 in $2q^2$ odd. This contradicts the fact that p^2 has an even number of factors of 2. Hence, we cannot have $p^2 = 2q^2$, and therefore $\sqrt{2} \notin \mathbb{Q}$. □

Theorem 2.27. *Consider the sets of rational numbers*

$$X = \{x \in \mathbb{Q} \mid x \geq 0 \text{ and } x^2 < 2\} \quad \text{and} \quad Y = \{y \in \mathbb{Q} \mid y > 0 \text{ and } y^2 > 2\}.$$

There are no rational numbers $a, b \in \mathbb{Q}$ such that $a = \sup X$ and $b = \inf Y$.

Proof. We prove the result concerning the supremum; the statement about the infimum follows analogously.

First, we claim that X has no maximum. Indeed, given any $x \in X$, choose $r \in \mathbb{Q}$ such that $r < 1$ and

$$0 < r < \frac{2 - x^2}{2x + 1}.$$

Note that $r < 1 \Rightarrow r^2 < r$, and we compute

$$(x+r)^2 = x^2 + 2xr + r^2 < x^2 + 2xr + r = x^2 + r(2x+1) < x^2 + (2 - x^2) = 2.$$

Therefore, $x + r \in X$, so x is not a maximum.

Similarly, one can show that Y has no minimum: given any $y \in Y$, there exists $r > 0$ such that $y - r \in Y$.

Also, observe that every element of X is strictly less than every element of Y. Indeed, if $x \in X$ and $y \in Y$, then $x^2 < 2 < y^2$, so

$$0 < y^2 - x^2 = (y - x)(y + x) \Rightarrow y - x > 0 \Rightarrow x < y.$$

Now suppose, for contradiction, that there exists $a \in \mathbb{Q}$ such that $a = \sup X$. Then $a \notin X$, otherwise it would be the maximum of X, contradicting the previous claim.

If $a \in Y$, then, since Y has no minimum, there exists $b \in Y$ such that $b < a$. But then $x < b < a$ for every $x \in X$, contradicting the fact that a is the least upper bound of X.

We conclude that $a \notin X$ and $a \notin Y$, so $a^2 = 2$, implying that $a = \sqrt{2}$, which contradicts Lemma 2.26, since $\sqrt{2} \notin \mathbb{Q}$. Therefore, $\sup X \notin \mathbb{Q}$, and the same argument applies to $\inf Y$. □

Since every ordered field contains $\mathbb{Q}$, it follows from the Theorem above that if there exists an ordered field K in which every nonempty subset that is bounded above has a supremum, then $a = \sup X$ exists in K, and this element must satisfy $a^2 = 2$.

In particular, such an ordered field K must contain an element whose square is 2. Hence, $\sqrt{2} \in K$, showing that K properly extends $\mathbb{Q}$ in this case.

Example 2.28. (A bounded set with no supremum) Let K be a non-Archimedean field. Then, by definition, $\mathbb{N} \subseteq K$ is bounded from above. Let $M \in K$ be an upper bound for $\mathbb{N}$. So $n + 1 \leq M$ for all $n \in \mathbb{N}$, but then $n \leq M - 1$ and $M - 1$ is also an upper bound. We conclude that if M is an upper bound, $M - 1$ is one as well, hence $\sup \mathbb{N}$ doesn't exists in K.

We say that an ordered field K is **complete**, if every nonempty bounded from above subset $X \subseteq K$ has a supremum in K. This motivates the following axiom (also called **the fundamental axiom of mathematical analysis**):

Axiom. There is a complete ordered field, represented by $\mathbb{R}$, called the field of real numbers.

Remark.

Notice that in a complete ordered field K, if $X \subseteq K$ is bounded from below then X has an infimum.

Remark.

From Example 2.28 we conclude that every complete ordered field is Archimedean.

Theorem 2.29. *If K, L are complete ordered fields, then there is an unique isomorphism $F : K \to L$.*

Proof. First we claim that given any complete ordered field F, there exists an unique isomorphism $f : \mathbb{R} \to F$. Let $1'$ denotes the unit in F and $0'$ its zero element. For $n \in \mathbb{N}$ set

$$n' = \underbrace{1' + 1' + \cdots + 1'}_{n \text{ times}} \quad \text{and} \quad (-n)' = -n'.$$

Define $f : \mathbb{R} \to F$ by

$$f(x) = \begin{cases} 0' & \text{if } x = 0 \\ \frac{p'}{q'} & \text{if } x = \frac{p}{q} \in \mathbb{Q} \setminus \{0\} \\ \sup\left\{\frac{p'}{q'} \in F\,;\ \frac{p}{q} < x\right\} & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

Let $x, y \in \mathbb{R}$. If $x, y \in \mathbb{Q}$ then it's easy to see that

$$f(x+y) = f(x) + f(y) \quad \text{and} \quad f(x \cdot y) = f(x) \cdot f(y).$$

Suppose $x \in \mathbb{R} \setminus \mathbb{Q}$ and $y = \frac{r}{s} \in \mathbb{Q}$. Then

$$\begin{aligned} f(x+y) &= \sup\left\{\frac{p'}{q'}\,;\ \frac{p}{q} < (x+y)\right\} \\ &= \sup\left\{\frac{p'}{q'}\,;\ \frac{p}{q} - \frac{r}{s} < x\right\} \\ &= \sup\left\{\frac{p'}{q'} - \frac{r'}{s'}\,;\ \frac{p}{q} - \frac{r}{s} < x\right\} + \frac{r'}{s'} \\ &= f(x) + f(y). \end{aligned}$$

Similarly,

$$\begin{aligned} f(x \cdot y) &= \sup\left\{\frac{p'}{q'}\,;\ \frac{p}{q} < (x \cdot y)\right\} \\ &= \sup\left\{\frac{p'}{q'}\,;\ \frac{p}{q}\frac{1}{y} < x\right\} \\ &= \sup\left\{\frac{p'}{q'}\frac{1'}{y'}\,;\ \frac{p}{q}\frac{1}{y} < x\right\} \cdot y' \\ &= f(x) \cdot f(y). \end{aligned}$$

The case where $x, y \notin \mathbb{Q}$ is left as an exercise.

We are left to prove that $f : \mathbb{R} \to F$ defines a bijection. It suffices to prove surjectivity (since every nontrivial field homomorphism is injective). Given $r \in F$, if $r = \frac{p'}{q'}$ then $f\left(\frac{p}{q}\right) = r$. Otherwise, consider the bounded set

$$X = \left\{\frac{p}{q} \in \mathbb{R}\,;\ \frac{p'}{q'} < r\right\}.$$

Then $x := \sup X$ satisfies $f(x) = r$.

We conclude that $f : \mathbb{R} \to F$ defines an isomorphism. We prove that it is unique. Suppose $g : \mathbb{R} \to F$ is another isomorphism. Then $H := f \circ g^{-1} : \mathbb{R} \to \mathbb{R}$ is an automorphism.

We claim H is the trivial automorphism. Since $H(1) = 1$, we already know that $H(x) = x$, if $x \in \mathbb{Q}$; moreover, H is increasing. Suppose $h \in \mathbb{R} \setminus \mathbb{Q}$ satisfies $H(h) \neq h$, say $H(h) < h$. Archimedes' principle guarantees the existence of a rational number $a \in \mathbb{Q}$ such that $H(h) < a < h$. However, this implies $a = H(a) < H(h)$, a contradiction. Therefore, H is the trivial automorphism. and $f = g$.

Now, let $f_1 : \mathbb{R} \to K$ and $f_2 : \mathbb{R} \to L$ be isomorphisms. We define $F : K \to L$ by

$$F(x) = (f_2 \circ f_1^{-1})(x).$$

Since the composition of bijections is a bijection and the composition of homomorphism is a homomorphism we conclude that F is an isomorphism. Uniqueness of F follows from the uniqueness of f_1 and f_2. □

The Theorem above says that, in some suitable sense, $\mathbb{R}$ is the only complete ordered field. Even though we assumed the existence of $\mathbb{R}$ through the fundamental axiom of mathematical analysis, it's possible to construct a complete ordered field explicitly:

Example 2.30. A **Dedekind cut** is a nonempty proper subset of the rationals, $A \subseteq \mathbb{Q}$, satisfying the following properties: A has no maximum element, and if $a \in A$, $b \in \mathbb{Q}$, and $b < a$, then $b \in A$.

Let $\mathcal{D}$ be the set of all Dedekind cuts. We define a field structure on $\mathcal{D}$ as follows. The zero element is

$$0' := \{x \in \mathbb{Q}\,;\, x < 0\}.$$

Similarly, the multiplicative identity is defined by

$$1' := \{x \in \mathbb{Q}\,;\, x < 1\}.$$

We define an order on $\mathcal{D}$ by $A < B$ if A is a proper subset of B. Hence, the set P of positive elements is defined by

$$P = \{A \in \mathcal{D}\,;\, A \text{ properly contains all negative rationals}\}.$$

The sum of two cuts is given by

$$A + B = \{a + b\,;\, a \in A,\ b \in B\}.$$

The definition of $A \cdot B$ is more elaborate. First, suppose $A, B \in P$. Then we set

$$A \cdot B = \{p \in \mathbb{Q}\,;\, p \leq a \cdot b \text{ for some } a \in A,\ b \in B,\ a, b > 0\}.$$

In general, we define

$$A \cdot B = \begin{cases} 0' & \text{if } A = 0' \text{ or } B = 0', \\ -(A \cdot -B) & \text{if } A > 0' \text{ and } B < 0', \\ -(-A \cdot B) & \text{if } A < 0' \text{ and } B > 0', \\ -A \cdot -B & \text{if } A < 0' \text{ and } B < 0'. \end{cases}$$

The set $\mathcal{D}$, together with the operations $+, \cdot$ and the order defined above, forms a complete ordered field. The proof of this fact is left as an exercise.

Notice that since complete ordered fields are unique up to isomorphism by Theorem 2.29, there exists an isomorphism $f : \mathbb{R} \to \mathcal{D}$.

The discussion above leads to the conclusion that although there is no rational number $x \in \mathbb{Q}$ such that $x^2 = 2$, there exists a positive real number $x \in \mathbb{R}$ satisfying $x^2 = 2$. We denote this number by $\sqrt{2}$. There is nothing special about the number 2; indeed, the argument generalizes to any $n \in \mathbb{N}$ that is not a perfect square. In such cases, we can similarly conclude that there exists a positive real number, denoted by $\sqrt{n}$, such that $(\sqrt{n})^2 = n$.

We can generalize even further by considering the n^{th} *root* of a natural number $m \in \mathbb{N}$, denoted by $\sqrt[n]{m}$. This is defined as the unique positive real number $x \in \mathbb{R}$ such that $x^n = m$.

The elements of the set $\mathbb{R} \setminus \mathbb{Q}$ are called **irrational numbers**. As we have just seen, there are many such numbers; for instance, all numbers of the form $\sqrt[n]{2}$, with $n \geq 2$, are irrational. In fact, we shall see next that irrational numbers are *everywhere* in a precise sense—as a subset of the real numbers.

A subset $X \subseteq \mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for every pair $a, b \in \mathbb{R}$ with $a < b$, there exists an element $x \in X$ such that $a < x < b$. In other words, X is dense in $\mathbb{R}$ if every open, non-degenerate interval (a, b) contains at least one point from X.

Example 2.31. Let $X = \mathbb{R} - \mathbb{Z}$. Then X is dense in $\mathbb{R}$. Indeed, every open interval (a, b) is an infinite set (since $\mathbb{R}$ is ordered). On the other hand, $\mathbb{Z} \cap (a, b)$ is finite, hence we can always find a number $x \notin \mathbb{Z}$ with $x \in (a, b)$.

Theorem 2.32. *The set of rational numbers, $\mathbb{Q}$, and the set of irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$, are both dense in $\mathbb{R}$.*

Proof. Let $(a, b) \subset \mathbb{R}$ be a non-degenerate open interval. Since $b - a > 0$, there exists a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$. Consider the set

$$X = \left\{ m \in \mathbb{Z} \ : \ \frac{m}{n} \geq b \right\}.$$

By the Archimedean property of $\mathbb{R}$, the set X is nonempty. Moreover, X is bounded below by $nb \in \mathbb{R}$. By the well-ordering principle, X has a smallest element, say $m_0 \in X$. By minimality of m_0, we have $m_0 - 1 \notin X$, hence

$$\frac{m_0 - 1}{n} < b.$$

We claim that $\frac{m_0-1}{n} > a$. Suppose not. Then

$$\frac{m_0 - 1}{n} \leq a < b \leq \frac{m_0}{n},$$

which implies

$$b - a \leq \frac{m_0}{n} - \frac{m_0 - 1}{n} = \frac{1}{n},$$

contradicting our choice of n. Therefore, the rational number $\frac{m_0-1}{n}$ lies in the interval (a, b), showing that $\mathbb{Q}$ is dense in $\mathbb{R}$.

To prove that $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$, we apply the same argument *mutatis mutandis*, replacing $\frac{1}{n}$ with an irrational number, such as $\frac{\sqrt{2}}{n}$. The rest of the proof proceeds identically, yielding an irrational number in (a, b). □

Theorem 2.33. *(The Nested Intervals Principle) Let $I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n \supseteq \ldots$ be a decreasing sequence of closed intervals of the form $I_n = [a_n, b_n]$. Then,*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset,$$

and more precisely,

$$\bigcap_{n=1}^{\infty} I_n = [a, b],$$

where $a = \sup a_n = \sup\{a_n \,:\, n \in \mathbb{N}\}$ and $b = \inf b_n = \inf\{b_n \,:\, n \in \mathbb{N}\}$.

Proof. By hypothesis, we have $I_n \supseteq I_{n+1}$ for all $n \in \mathbb{N}$, which implies the following chain of inequalities:

$$a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots \leq b_n \leq \ldots \leq b_2 \leq b_1.$$

In particular, the sequence (a_n) is increasing and bounded above by b_1, so the supremum

$$a := \sup\{a_n : n \in \mathbb{N}\} \in \mathbb{R}$$

is well defined. Similarly, since (b_n) is decreasing and bounded below by a_1, the infimum

$$b := \inf\{b_n : n \in \mathbb{N}\} \in \mathbb{R}$$

is also well defined.

Since each b_n is an upper bound for the set $\{a_k \,:\, k \in \mathbb{N}\}$, it follows that $a \leq b_n$ for all $n \in \mathbb{N}$. Hence,

$$a_n \leq a \leq b_n \quad \text{for all } n \in \mathbb{N}.$$

By a similar argument, $a_n \leq b \leq b_n$ for all $n \in \mathbb{N}$. Therefore, the closed interval $[a, b]$ is contained in every I_n, i.e.,

$$[a, b] \subseteq I_n \quad \text{for all } n \in \mathbb{N}.$$

We now show that no point outside $[a, b]$ lies in the intersection. Suppose $x < a$. Then, since $a = \sup a_n$, there exists $n_0 \in \mathbb{N}$ such that $x < a_{n_0}$, and thus $x \notin I_{n_0}$, implying $x \notin \bigcap_{n=1}^{\infty} I_n$.

Similarly, if $x > b$, then since $b = \inf b_n$, there exists $n_1 \in \mathbb{N}$ such that $x > b_{n_1}$, so $x \notin I_{n_1}$ and again $x \notin \bigcap_{n=1}^{\infty} I_n$.

We conclude that

$$\bigcap_{n=1}^{\infty} I_n = [a, b].$$

□

Theorem 2.34. *The set of real numbers* $\mathbb{R}$ *is uncountable.*

Proof. Let $X = \{x_1, x_2, \dots\} \subseteq \mathbb{R}$ be a countable subset of $\mathbb{R}$, which we know exists by Theorem 1.58. We claim there is always an $x \in \mathbb{R}$ such that $x \notin X$. Pick a closed interval I_1 not containing x_1, this is possible since $\mathbb{R}$ is infinite. Proceed by induction, after setting I_n not containing x_n, we select $I_{n+1} \subseteq I_n$ as a closed interval which doesn't contain x_{n+1}. Proceeding this way, we construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \dots I_n \supseteq \dots$. Therefore, by Theorem 2.33, there is at least one $x \in \mathbb{R}$ that is not in X. □

Corollary 2.35. *Any non-degenerate open interval* $(a, b) \subset \mathbb{R}$ *is uncountable.*

Proof. Define the function $f : (0, 1) \to (a, b)$ by

$$f(x) = (b - a)x + a.$$

This function is clearly bijective. Therefore, it suffices to show that the interval $(0, 1)$ is uncountable.

Suppose, for contradiction, that $(0, 1)$ is countable. Then the set $(0, 1] = (0, 1) \cup \{1\}$ is also countable as the union of two countable sets. Moreover, for each integer $n \in \mathbb{Z}$, the interval

$$(n, n + 1] = \{x \in \mathbb{R} : n < x \leq n + 1\}$$

is a translation of $(0, 1]$, and thus countable.

Since

$$\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n + 1],$$

we would have that $\mathbb{R}$ is a countable union of countable sets, hence countable. This contradicts the fact that $\mathbb{R}$ is uncountable.

Therefore, $(0, 1)$, and hence (a, b), is uncountable. □

Corollary 2.36. *The set of irrational numbers* $\mathbb{R} \setminus \mathbb{Q}$ *is uncountable.*

Proof. Suppose, for contradiction, that $\mathbb{R} \setminus \mathbb{Q}$ is countable. Since the set of rational numbers $\mathbb{Q}$ is also countable, it would follow that

$$\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$$

is a countable union of countable sets, and therefore countable. This contradicts the fact that $\mathbb{R}$ is uncountable. Hence, $\mathbb{R} \setminus \mathbb{Q}$ must be uncountable. □

Exercises

1. Let K, L be fields. A function $f : K \to L$ is called homomorphism when $f(x + y) = f(x) + f(y)$ and $f(x \cdot y) = f(x) \cdot f(y)$, for any $x, y \in K$. Given a homomorphism $f : K \to L$ show that $f(0) = 0$. Also, show that only one of the following happens: $f(x) = 0, \forall x \in K$ or $f(1) = 1$ and f is injective.
2. Given a homomorphism $f : \mathbb{Q} \to \mathbb{Q}$. Show that only one of the following happens: $f(x) = 0, \forall x \in \mathbb{Q}$ or $f(x) = x, \forall x \in \mathbb{Q}$.
3. Explain why $\mathbb{Z}$, with its usual operations, is not a field.
4. Let K be an ordered field and $a, b \in K$. Show that $a^2 + b^2 = 0 \Leftrightarrow a = b = 0$.
5. Let $\mathcal{F}(X; K)$ denotes the set of all functions between X and K. Given $f, g \in \mathcal{F}(X; K)$, define the following operations on set the set $\mathcal{F}(X; K)$: $(f + g)(x) = f(x) + g(x)$ and $(f \cdot g)(x) = f(x) \cdot g(x)$. Is $\mathcal{F}(X; K)$ a field?
6. Let x, y be positive elements of an ordered field K. Show that
$$x < y \Leftrightarrow x^{-1} > y^{-1}$$
7. Let $x \in K$ be a nonzero element in a ordered field K and $n \in \mathbb{N}$. Show that
$$(1 + x)^{2n} > 1 + 2n \cdot x$$
8. Let K be an ordered field and $a, x \in K$. If a and $a + x$ are positive, show that
$$(a + x)^n \geq a^n + n \cdot a^{n-1} \cdot x$$
.
9. Given an ordered field K, show the following are equivalent:
   a. K is Archimedean;
   b. $\mathbb{Z}$ is unbounded from below and from above;
   c. $\mathbb{Q}$ is unbounded from below and from above.
10. Given an ordered field K, show that K is Archimedean $\Leftrightarrow \forall \epsilon > 0 \in K, \exists n \in \mathbb{N}$ such that $\frac{1}{2^n} < \epsilon$.
11. Let $a > 1$ be an element of an Archimedean field K. Consider the function $f : \mathbb{Z} \to K$, given by $f(n) = a^n$. Show the following:
    a. $f(\mathbb{Z})$ is not bounded from above;

b. $\inf f(\mathbb{Z}) = 0$.

12. Let $a, b, c, d \in \mathbb{Q}$. Show that

$$a + b\sqrt{2} = c + d\sqrt{2} \Leftrightarrow a = c \text{ and } b = d.$$

13. Let $a, b \in \mathbb{Q}$ be positive numbers. Show that

$$\sqrt{a} + \sqrt{b} \text{ is rational} \Leftrightarrow \text{ both } \sqrt{a} \text{ and } \sqrt{b} \text{ are rational.}$$

14. Let $X = \{\frac{1}{n}\,;\, n \in \mathbb{N}\}$. Show that $\inf X = 0$.
15. Let $A \subseteq B \subseteq \mathbb{R}$ be nonempty bounded sets. Show that

$$\inf B \le \inf A \le \sup A \le \sup B.$$

16. Let $A \subseteq \mathbb{R}$ be a bounded nonempty set. Show that

$$\sup -A = -\inf A.$$

17. Let $A \subseteq \mathbb{R}$ be a bounded nonempty set and $c > 0$, show that

$$\sup c \cdot A = c \cdot \sup A$$

.

18. Let $A, B \subseteq \mathbb{R}$ be bounded nonempty sets. Show that

$$\sup(A + B) = \sup A + \sup B;$$

and similarly, show that

$$\sup(A \cdot B) = \sup A \cdot \sup B,$$

where $A \cdot B = \{x \cdot y\,;\, x \in A, y \in B\}$.

19. Let $p > 1$ be a natural number. Show the set

$$X = \left\{\frac{m}{p^n}\,;\, m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\right\}$$

is dense in $\mathbb{R}$.

20. A number $r \in \mathbb{R}$ is said to be **algebraic** if it is a root of a polynomial $p(x) \in \mathbb{Z}[x]$ with integral coefficients.

    a. Show that the set of all polynomials with integral coefficients, $\mathbb{Z}[x]$, is countable.
    b. Show that the set of all algebraic numbers is countable and dense in $\mathbb{R}$.

21. Let $X = \mathbb{R} - A$, where A is a countable subset of $\mathbb{R}$. Show that for each open interval (a, b), the intersection $X \cap (a, b)$ is uncountable. In particular, X is dense in $\mathbb{R}$.

22. A number $r \in \mathbb{R}$ is said to be **transcendental** if it's not algebraic. Show that the set of all transcendental numbers is uncountable and dense in $\mathbb{R}$.
23. Show that the set of algebraic numbers, usually denoted by $\overline{\mathbb{Q}}$, can be given a field structure. *This exercise assumes knowledge of Abstract algebra, you may skip it if you want.*
24. Give an Example of open bounded nested intervals whose intersection is empty.
25. Show that the set $\mathcal{D}$ of all Dedekind cuts (see Example 2.30) is a complete ordered field.
26. Let X, Y be nonempty sets and $f : X \times Y \to \mathbb{R}$ a bounded function, i.e. $|f(x)| \leq c$. Let $f_1(x) = \sup\{f(x, y); y \in Y\}$ and $f_2(y) = \sup\{f(x, y); x \in X\}$. Show that
$$\sup_{x \in X} f_1(x) = \sup_{y \in Y} f_2(y).$$
In other words, sup commutes with itself:
$$\sup_x(\sup_y f(x, y)) = \sup_y(\sup_x f(x, y))$$
27. Generalize the exercise above and show that
$$\sup_y(\inf_x f(x, y)) \leq \inf_x(\sup_y f(x, y))$$
28. Let $x, y \in \mathbb{R}$ be positive numbers. Show that $\sqrt{x \cdot y} \leq \frac{x+y}{2}$
29. Show that the function $f : \mathbb{R} \to (-1, 1)$ defined by $f(x) = \frac{x}{\sqrt{1+x^2}}$ is a bijection.

Chapter 3
Sequences and Series of real numbers

3.1 Sequences

A **sequence of real numbers**, denoted by $x_n := x(n)$, is a function $x : \mathbb{N} \to \mathbb{R}$. There is no universal notation for a sequence, but common ones include

$$\{x_n\}_{n\in\mathbb{N}}, \ \mathbf{x}_n, \ \{x_1, x_2, \ldots\}, \ (x_n).$$

A sequence x_n is *bounded* if there exist $a, b \in \mathbb{R}$ with

$$a \le x_n \le b \quad (n \in \mathbb{N}).$$

Equivalently, $x(\mathbb{N}) \subseteq [a, b]$. A sequence is *unbounded* if it is not bounded.

It is *bounded above* if $x_n \le b$ for some $b \in \mathbb{R}$, and *bounded below* if $a \le x_n$ for some $a \in \mathbb{R}$. A sequence is bounded $\iff$ it is both bounded above and bounded below.

Let $K \subseteq \mathbb{N}$ be infinite. Then K is countable, so there exists a bijection $b : \mathbb{N} \to K$, $k \mapsto n_k$. For any sequence $x : \mathbb{N} \to \mathbb{R}$, the sequence

$$x_{n_k} := x \circ b : \mathbb{N} \to \mathbb{R}$$

is called a **subsequence** of x_n.

Example 3.1. If $K = \{n \in \mathbb{N} : n \text{ even}\}$ and $b(k) = 2k$, then $x_{n_k} = x_{2k}$ is a subsequence of x_n. For instance, if $x_n = (-1)^n$, then $x_{2k} = 1$ for all k.

Every subsequence of a bounded sequence is bounded.

A sequence is *nondecreasing* if $x_n \le x_{n+1}$ for all n, and *increasing* if $x_n < x_{n+1}$ for all n. Similarly, it is *nonincreasing* if $x_n \ge x_{n+1}$ for all n, and *decreasing* if $x_n > x_{n+1}$ for all n.

A sequence that is nondecreasing, nonincreasing, increasing, or decreasing is called **monotone**.

Lemma 3.2. *A monotone sequence x_n is bounded $\iff$ it has a bounded subsequence.*

Proof. The forward direction is immediate. For the converse, suppose x_{n_k} is a bounded subsequence of a monotone sequence x_n. Assume x_n is nondecreasing. Then $x_{n_k} \leq b$ for some $b \in \mathbb{R}$. For any n, choose $n_k > n$. Then $x_n \leq x_{n_k} \leq b$, so x_n is bounded. □

Example 3.3. The sequence $x_n = 1$, i.e. $\{1, 1, 1, \ldots\}$, is constant, bounded, nonincreasing, and nondecreasing.

Example 3.4. The sequence $x_n = n$, i.e. $\{1, 2, 3, \ldots\}$, is an unbounded increasing sequence.

Example 3.5. The sequence $x_n = \frac{1}{n}$, i.e. $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, is a bounded decreasing sequence, since $0 < x_n \leq 1$.

Example 3.6. The sequence $x_n = 1 + (-1)^n$, i.e. $\{0, 2, 0, 2, \ldots\}$, is bounded but not monotone.

Example 3.7. The sequence

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

is increasing and bounded, since

$$0 < x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}} < 3.$$

The sequence $y_n = (1 + \frac{1}{n})^n$ is related: by the binomial Theorem, $y_n \leq x_n$, hence $0 < y_n < 3$.

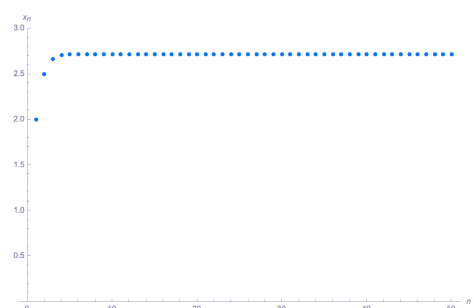


Fig. 3.1: $y_n = (1 + \frac{1}{n})^n$

Example 3.8. Let $x_1 = 0$, $x_2 = 1$, and define $x_{n+2} = x_{n+1} + x_n$ for $n \geq 1$. It is easy to check that $0 \leq x_n \leq 1$. A computation shows

$$x_{2n} = 1 - \left(\frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}\right), \qquad x_{2n+1} = \frac{1}{2}\left(1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^{n-1}}\right).$$

Thus x_n is bounded but not monotone.

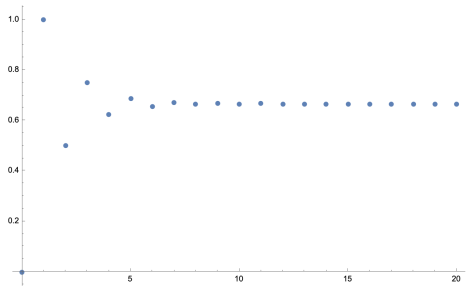


Fig. 3.2: $x_{n+2} = x_{n+1} + x_n$

Example 3.9. Let $a \in \mathbb{R}$ with $0 < a < 1$. The sequence

$$x_n = 1 + a + a^2 + \cdots + a^n = \frac{1 - a^{n+1}}{1 - a}$$

is increasing (since $a > 0$) and bounded, since $0 < x_n \leq \frac{1}{1-a}$.

Example 3.10. Consider the sequence $\{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \ldots\}$ given by $x_n = \sqrt[n]{n}$. It increases for $n = 1, 2$. We claim that for $n \geq 3$ it is decreasing. Indeed, $x_{n+1} < x_n$ is equivalent to $(n+1)^n < n^{n+1}$, i.e.

$$\left(1 + \frac{1}{n}\right)^n < n,$$

which holds for $n \geq 3$ by Example 3.7. Hence x_n is bounded.

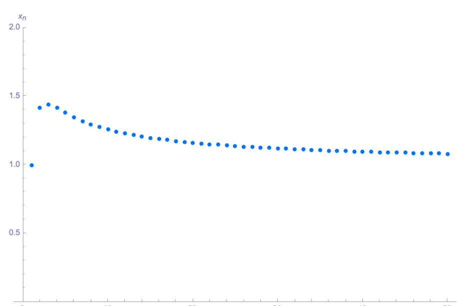


Fig. 3.3: $x_n = \sqrt[n]{n}$

3.2 The limit of a sequence

Informally, to say that $a \in \mathbb{R}$ is the limit of the sequence x_n means that the terms of the sequence become arbitrarily close to a as n grows large. More precisely:

$$\lim_{n\to\infty} x_n = a \quad := \quad \forall\epsilon > 0 \;\; \exists n_0 \in \mathbb{N} \;\; (n > n_0 \;\Rightarrow\; |x_n - a| < \epsilon).$$

In words: *"The limit of the sequence x_n is a if, for every $\epsilon > 0$, no matter how small, there exists an index n_0 such that $|x_n - a| < \epsilon$ whenever $n > n_0$."*

Equivalently, every open interval $(a - \epsilon, a + \epsilon)$ centered at a contains all but finitely many terms of the sequence.

Remark. *It is common practice to omit "$n \to \infty$" and write simply* $\lim x_n$.

When $\lim x_n = a$, we say that x_n *converges* to a, also written

$$x_n \to a,$$

and call x_n *convergent*. If x_n is not convergent, we say it is *divergent*, i.e. there is no $a \in \mathbb{R}$ with $\lim x_n = a$.

Theorem 3.11. *(Uniqueness of limits) If* $\lim x_n = a$ *and* $\lim x_n = b$*, then* $a = b$.

Proof. Suppose $\lim x_n = a$ and $b \neq a$. It suffices to show that $\lim x_n \neq b$. Let $\epsilon = \frac{|b-a|}{2}$. Since $\lim x_n = a$, there exists n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$. Hence, for $n > n_0$ we have $x_n \notin (b - \epsilon, b + \epsilon)$, and therefore $\lim x_n \neq b$. □

Theorem 3.12. *If* $\lim x_n = a$*, then every subsequence* x_{n_k} *also satisfies* $\lim x_{n_k} = a$.

Proof. Given $\epsilon > 0$, choose n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$. Since $n_k > n_0$ implies $|x_{n_k} - a| < \epsilon$, the same n_0 works for the subsequence. □

Corollary 3.13. *If* $k \in \mathbb{N}$ *and* $\lim x_n = a$*, then* $\lim x_{n+k} = a$*, since* x_{n+k} *is a subsequence of* x_n.

In other words, Corollary 3.13 says that the limit of a sequence does not change if we omit finitely many terms.

Theorem 3.14. *Every convergent sequence* x_n *is bounded.*

Proof. Suppose $\lim x_n = a$. Then there exists n_0 such that $n > n_0 \Rightarrow x_n \in (a - 1, a + 1)$. Let

$$M = \max\{|x_1|, \ldots, |x_{n_0}|, |a - 1|, |a + 1|\}.$$

Then $|x_n| \leq M$ for all n, so x_n is bounded. □

Example 3.15. The sequence $\{0, 1, 0, 1, 0, 1, \ldots\}$ is not convergent by Theorem 3.12, since it has subsequences with different limits: $x_{2n} = 1$ and $x_{2n-1} = 0$. This shows that a bounded sequence need not be convergent, i.e. the converse of Theorem 3.14 fails.

Theorem 3.16. *(Monotone Convergence Theorem) Every bounded monotone sequence is convergent.*

Proof. Assume x_n is nondecreasing (the other cases are analogous). Since x_n is bounded, $a = \sup\{x_n\}$ is well defined. Given $\epsilon > 0$, choose n_0 such that $a - \epsilon < x_{n_0}$. By monotonicity, $a - \epsilon < x_n$ for all $n \geq n_0$. Clearly $x_n \leq a$, so $a - \epsilon < x_n < a + \epsilon$ for $n > n_0$. Thus $\lim x_n = a$. □

Corollary 3.17. *If a monotone sequence x_n has a convergent subsequence, then x_n is convergent.*

Proof. Suppose x_n is increasing (the other cases are analogous) and that x_{n_k} converges. Then x_{n_k} is bounded, say $|x_{n_k}| \leq M$ for all k. Given any $n \in \mathbb{N}$, we can choose an index k_0 such that $n < n_{k_0}$. Since x_n is increasing, we have $x_n < x_{n_{k_0}} \leq M$. Thus x_n is bounded. By Theorem 3.16, every bounded monotone sequence converges, and therefore x_n is convergent. □

Example 3.18. Every constant sequence $x_n = k \in \mathbb{R}$ is convergent and $\lim x_n = k$.

Example 3.19. The sequence $\{1, 2, 3, 4, \ldots\}$ is divergent because it is unbounded.

Example 3.20. The sequence $\{1, -1, 1, -1, \ldots\}$ is divergent because it has two subsequences converging to different values, namely $x_{2n} = 1$ and $x_{2n-1} = -1$.

Example 3.21. The sequence $x_n = \frac{1}{n}$ is convergent with $\lim x_n = 0$. Indeed, since $\mathbb{R}$ is Archimedean, given $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \epsilon$. Then for all $n > n_0$, we have $\frac{1}{n} < \epsilon$.

Example 3.22. Let $0 < a < 1$. The sequence $x_n = a^n$ is monotone decreasing and bounded below by 0, hence convergent. Notice that $\lim x_n = 0$ in this case.

Theorem 3.23. *If* $\lim x_n = 0$ *and y_n is a bounded sequence, then*

$$\lim(x_n \cdot y_n) = 0.$$

Proof. Since y_n is bounded, there exists $c > 0$ such that $|y_n| < c$ for all n. Let $\epsilon > 0$ be given. Since $\lim x_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \implies |x_n| < \frac{\epsilon}{c}$. Then for all $n > n_0$, we have

$$|x_n y_n| \leq |x_n| \cdot |y_n| < \frac{\epsilon}{c} \cdot c = \epsilon,$$

which proves that $\lim(x_n y_n) = 0$. □

Example 3.24. It follows from the Theorem above that

$$\lim_{n\to\infty} \frac{\sin n}{n} = 0,$$

since $|\sin n| \le 1$ and $\frac{1}{n} \to 0$.

Theorem 3.25. *Let* $\lim x_n = a$ *and* $\lim y_n = b$. *Then*

1. $\lim(x_n + y_n) = a + b$ *and* $\lim(x_n - y_n) = a - b$;
2. $\lim(x_n \cdot y_n) = ab$;
3. *If* $b \neq 0$, *then* $\lim \dfrac{x_n}{y_n} = \dfrac{a}{b}$.

Proof. **(1)** Let $\epsilon > 0$. Since $\lim x_n = a$ and $\lim y_n = b$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$n > N_1 \implies |x_n - a| < \frac{\epsilon}{2}, \qquad n > N_2 \implies |y_n - b| < \frac{\epsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$. For $n > N$,

$$|(x_n + y_n) - (a + b)| \le |x_n - a| + |y_n - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

so $\lim(x_n + y_n) = a + b$. The difference case is identical, since

$$|(x_n - y_n) - (a - b)| \le |x_n - a| + |y_n - b|.$$

(2) Convergent sequences are bounded, so there exists $M > 0$ with $|x_n| \le M$ and $|y_n| \le M$ for all n (one may take different bounds for x_n and y_n; use the larger). Let $\epsilon > 0$. Choose N_1 with $n > N_1 \implies |x_n - a| < \delta$ and N_2 with $n > N_2 \implies |y_n - b| < \delta$, where we will specify $\delta > 0$ shortly. For $n > \max\{N_1, N_2\}$,

$$|x_n y_n - ab| = |x_n y_n - a y_n + a y_n - ab| \le |x_n - a|\ |y_n| + |a|\ |y_n - b|.$$

Using the bound $|y_n| \le M$, we get

$$|x_n y_n - ab| \le M|x_n - a| + |a||y_n - b|.$$

Choose $\delta > 0$ so that $M\delta + |a|\delta < \epsilon$ (for example $\delta = \epsilon/(M + |a|)$). Then for $n > \max\{N_1, N_2\}$ the right-hand side is $< \epsilon$, proving $\lim x_n y_n = ab$.

(Alternate standard decomposition: $x_n y_n - ab = (x_n - a)(y_n - b) + a(y_n - b) + b(x_n - a)$, and each term tends to 0.)

(3) Assume $b \neq 0$. Since $y_n \to b$, there exists N_0 such that for all $n > N_0$ we have $y_n \in (b - \frac{b}{2}, b + \frac{b}{2})$, in particular, $y_n > 0$ for those values of n. Thus the sequence y_n is bounded away from 0 eventually, so the quotients are defined for all large n.

Let $\epsilon > 0$. Pick N_1 with $n > N_1 \implies |x_n - a| < \delta_1$ and N_2 with $n > N_2 \implies |y_n - b| < \delta_2$, where we will choose $\delta_1, \delta_2 > 0$ below. For $n > \max\{N_0, N_1, N_2\}$,

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| = \left|\frac{x_n b - a y_n}{y_n b}\right| = \frac{|b(x_n - a) - a(y_n - b)|}{|y_n||b|} \leq \frac{|b|\,|x_n - a| + |a|\,|y_n - b|}{|y_n||b|}.$$

Using $|y_n| \geq \frac{|b|}{2}$ for such n, we obtain

$$\left|\frac{x_n}{y_n} - \frac{a}{b}\right| \leq \frac{2}{|b|^2}\big(|b|\,|x_n - a| + |a|\,|y_n - b|\big) = \frac{2}{|b|}\,|x_n - a| + \frac{2|a|}{|b|^2}\,|y_n - b|.$$

Now choose $\delta_1, \delta_2 > 0$ so that

$$\frac{2}{|b|}\,\delta_1 + \frac{2|a|}{|b|^2}\,\delta_2 < \epsilon,$$

and pick N_1, N_2 accordingly. For $n > \max\{N_0, N_1, N_2\}$ the left-hand side is $< \epsilon$. Hence $\lim \dfrac{x_n}{y_n} = \dfrac{a}{b}$. □

Example 3.26. Let $a \in \mathbb{R}$ be a positive number. The sequence $x_n = \sqrt[n]{a}$ is bounded and monotone, hence converges. We claim that

$$\lim_{n\to\infty} \sqrt[n]{a} = 1.$$

Indeed, let $L = \lim \sqrt[n]{a}$ and consider the subsequence $y_n = x_{n(n+1)}$. Then

$$L = \lim y_n = \lim a^{\frac{1}{n(n+1)}} = \lim a^{\frac{1}{n} - \frac{1}{n+1}} = \frac{\lim a^{\frac{1}{n}}}{\lim a^{\frac{1}{n+1}}} = 1.$$

Example 3.27. Consider the sequence $x_n = \sqrt[n]{n}$ from Example 3.10. From $n \geq 3$ onward, the sequence is decreasing (and bounded), hence convergent. We claim that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Let $L = \lim \sqrt[n]{n}$ and consider the subsequence $y_n = x_{2n} = \sqrt[2n]{2n}$. Then

$$L^2 = \lim y_n \cdot y_n = \lim \sqrt[n]{2n} = \lim\big(\sqrt[n]{2} \cdot \sqrt[n]{n}\big) = 1 \cdot L = L.$$

Thus $L = 0$ or $L = 1$. Since $x_n \geq 1$ for all n, we conclude $L = 1$.

Example 3.28. The sequence

$$x_n = 1 + 1 + \tfrac{1}{2!} + \ldots + \tfrac{1}{n!}$$

is increasing. It is also bounded since

$$2 \leq x_n \leq 1 + 1 + \tfrac{1}{2} + \ldots + \tfrac{1}{2^{n-1}} < 3.$$

Hence x_n converges. Its limit, denoted by e, is called the *Euler constant*. Our discussion shows that $2 < e < 3$. The increasing sequence $y_n = (1 + \frac{1}{n})^n$ is also related to e, since $y_n \leq x_n$ and $\lim y_n = \lim x_n = e$.

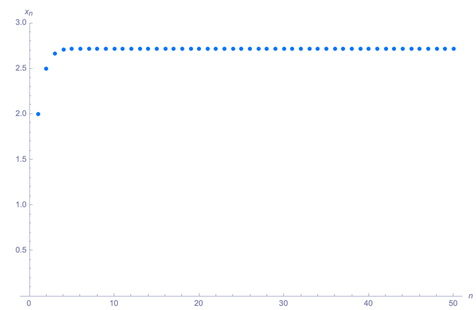


Fig. 3.4: $x_n = 1 + 1 + \frac{1}{2!} + \ldots + \frac{1}{n!}$

Theorem 3.29. *If* $\lim x_n = a$ *and* $a > 0$*, then there exists* $n_0 \in \mathbb{N}$ *such that* $x_n > 0$ *for all* $n > n_0$*. An analogous statement holds if* $a < 0$*, namely that eventually* $x_n < 0$*.*

Proof. Since $\lim x_n = a$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \implies |x_n - a| < \frac{a}{2}$. In particular, this implies $x_n > \frac{a}{2} > 0$ for $n > n_0$. The case $a < 0$ follows similarly. □

Corollary 3.30. *If* x_n *and* y_n *are convergent sequences with* $x_n \leq y_n$ *for all* n*, then* $\lim x_n \leq \lim y_n$*.*

Proof. Let

$$\lim x_n = a \quad \text{and} \quad \lim y_n = b.$$

We want to show that $a \leq b$.

Suppose, for the sake of contradiction, that $a > b$. Then $a - b > 0$. By Theorem 3.29, since $\lim(x_n - y_n) = a - b > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n - y_n > 0 \quad \text{for all } n > n_0,$$

which implies

$$x_n > y_n \quad \text{for all } n > n_0,$$

a contradiction. □

Corollary 3.31. *If* x_n *is convergent and* $x_n \geq a \in \mathbb{R}$ *for all* n*, then* $\lim x_n \geq a$*.*

Proof. Take $x_n = a$ in Corollary 3.30. □

Theorem 3.32. *(Squeeze Theorem) If* $x_n \le y_n \le z_n$ *for all n, and*

$$\lim x_n = \lim z_n = L,$$

then $\lim y_n = L$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim x_n = L$, there exists $n_1 \in \mathbb{N}$ such that

$$|x_n - L| < \varepsilon \quad \text{for all } n > n_1.$$

Similarly, since $\lim z_n = L$, there exists $n_2 \in \mathbb{N}$ such that

$$|z_n - L| < \varepsilon \quad \text{for all } n > n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then for all $n > n_0$, we have

$$L - \varepsilon < x_n \le y_n \le z_n < L + \varepsilon,$$

which implies

$$|y_n - L| < \varepsilon \quad \text{for all } n > n_0.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $\lim y_n = L$. □

3.3 lim inf x_n and lim sup x_n

A number $a \in \mathbb{R}$ is an *accumulation point* of a sequence x_n if there exists a subsequence x_{n_k} such that

$$\lim_{k\to\infty} x_{n_k} = a.$$

Theorem 3.33. *A number* $a \in \mathbb{R}$ *is an accumulation point of the sequence* x_n *if and only if*

$$\forall \varepsilon > 0, \text{ there are infinitely many } n \in \mathbb{N} \text{ such that } x_n \in (a - \varepsilon, a + \varepsilon).$$

Proof. The forward implication follows directly from the definition. For the converse, let $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{k}, \ldots$. Since there are infinitely many n with $x_n \in (a - \frac{1}{k}, a + \frac{1}{k})$, we can select indices $n_1 < n_2 < \cdots < n_k < \ldots$ such that

$$|x_{n_k} - a| < \frac{1}{k} \quad \text{for each } k \in \mathbb{N}.$$

By construction, $\lim_{k\to\infty} x_{n_k} = a$, so a is an accumulation point. □

Example 3.34. If $\lim x_n = a$, then x_n has exactly one accumulation point, namely a. This follows directly from Theorem 3.12.

Example 3.35. Consider the sequence $\{0, 1, 0, 2, 0, 3, \dots\}$. It diverges, but has 0 as an accumulation point due to the constant subsequence $x_{2n-1} = 0$. Similarly, the sequence $\{1, -1, 1, -1, \dots\}$ has two accumulation points: -1 and 1. The sequence $\{1, 2, 3, 4, 5, \dots\}$ is divergent and has no accumulation points.

Example 3.36. By Theorem 2.32, every real number $r \in \mathbb{R}$ is an accumulation point of some sequence of rational numbers.

We shall see below that every bounded sequence has at least one accumulation point, and converges if and only if it has a unique accumulation point.

Let x_n be a bounded sequence, so that $m \le x_n \le M$ for all n, with $m, M \in \mathbb{R}$. Define the sets

$$X_n := \{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

Then $X_{n+1} \subseteq X_n \subseteq [m, M]$ for all n. Set

$$a_n := \inf X_n, \quad b_n := \sup X_n.$$

The sequences (a_n) and (b_n) are monotone and bounded:

$$m \le a_1 \le a_2 \le \cdots \le a_n \le \cdots \le b_n \le \cdots \le b_2 \le b_1 \le M,$$

so their limits exist. Define

$$\liminf x_n := \lim_{n\to\infty} a_n, \quad \limsup x_n := \lim_{n\to\infty} b_n.$$

It is immediate that

$$\liminf x_n \le \limsup x_n.$$

Example 3.37. Consider the sequence $x_n = \{0, 1, 0, 1, 0, 1, \dots\}$. Using the notation above, $a_n \equiv 0$ and $b_n \equiv 1$, so $\liminf x_n = 0$ and $\limsup x_n = 1$.

Theorem 3.38. *Let x_n be a bounded sequence. Then* $\liminf x_n$ *is the smallest accumulation point, and* $\limsup x_n$ *is the greatest accumulation point.*

Proof. We prove the claim for $\liminf x_n$; the proof for $\limsup x_n$ is analogous.

Let $a := \liminf x_n = \lim a_n$. Given any $\varepsilon > 0$, choose n_0 such that $a - \varepsilon < a_{n_0} \le a < a + \varepsilon$. Since $a_{n_0} = \inf\{x_{n_0}, x_{n_0+1}, \dots\}$, there exists $n_1 \ge n_0$ such that $a - \varepsilon < x_{n_1} < a + \varepsilon$. Repeating this process produces a subsequence converging to a, so a is an accumulation point.

To see minimality, let $c < a$. Then there exists n_0 such that $c < a_{n_0} \le x_n$ for all $n \ge n_0$. Choosing $\varepsilon = a_{n_0} - c$, the interval $(c - \varepsilon, c + \varepsilon)$ contains no x_n for $n > n_0$. By Theorem 3.33, c is not an accumulation point. □

Example 3.39. ByTheorem 3.38, it follows that the sequence $x_n = \sin n$ satisfies $\liminf x_n = -1$ and $\limsup x_n = 1$.

Corollary 3.40. *(Bolzano–Weierstrass Theorem) Every bounded sequence x_n has a convergent subsequence.*

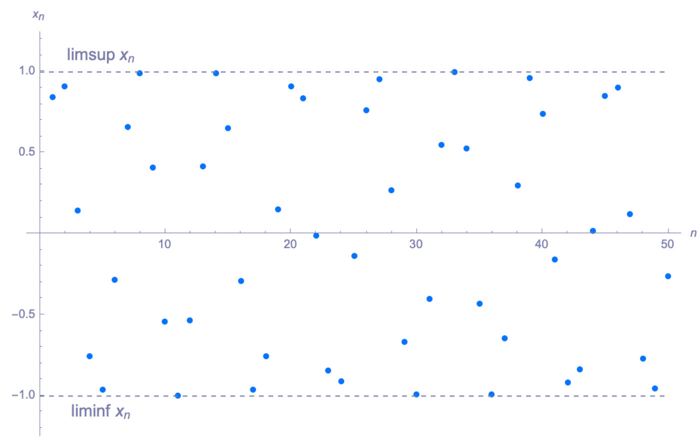


Fig. 3.5: $x_n = \sin n$

Proof. Since x_n is bounded, $a = \liminf x_n$ is well defined and is an accumulation point. Hence, there exists a subsequence converging to a. □

Corollary 3.41. *A sequence x_n is convergent if and only if* $\liminf x_n = \limsup x_n$.

Proof. If x_n converges, all subsequences converge to the same limit, so $\liminf x_n = \limsup x_n = \lim x_n$. Conversely, suppose $a = \liminf x_n = \limsup x_n$. Then for any $\varepsilon > 0$, there exists n_0 such that $a - \varepsilon < x_n < a + \varepsilon$ for all $n > n_0$, so $x_n \to a$. □

Corollary 3.42. *If $c < \liminf x_n$, then there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \implies c < x_n$. Similarly, if $c > \limsup x_n$, then there exists $n_1 \in \mathbb{N}$ such that $n > n_1 \implies c > x_n$.*

Proof. We prove the first statement; the second is analogous.

Let $a = \liminf x_n$ be the smallest accumulation point of (x_n), and assume $c < a$. Suppose, for contradiction, that for every $n \in \mathbb{N}$ there exists $m > n$ with $x_m \leq c$. Then we can construct a subsequence (x_{n_k}) with $x_{n_k} \leq c$ for all k.

Since (x_n) is bounded, (x_{n_k}) has a convergent subsequence with limit $b \leq c < a$. But b is an accumulation point, contradicting the minimality of a. □

A sequence x_n is called a **Cauchy sequence** if, given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$,

$$|x_n - x_m| < \varepsilon.$$

In other words, a Cauchy sequence is a sequence whose terms x_n become arbitrarily close to each other for sufficiently large n. It is reasonable to expect that a sequence with this property converges, and indeed this is true, as the theorem below shows.

Theorem 3.43. *Every Cauchy sequence is convergent.*

Proof. The proof follows directly from the two lemmas below.

Lemma 3.44. *Every Cauchy sequence is bounded.*

Proof. Let (x_n) be a Cauchy sequence. By definition, there exists $n_0 \in \mathbb{N}$ such that

$$m, n > n_0 \implies |x_n - x_m| < 1.$$

Fix x_{n_0} and define

$$M := \max\left\{|x_1|, |x_2|, \ldots, |x_{n_0}|, |x_{n_0} - 1|, |x_{n_0} + 1|\right\}.$$

Then for all n, $|x_n| \leq M$, and hence the sequence (x_n) is bounded. □

Lemma 3.45. *If a Cauchy sequence (x_n) has a convergent subsequence (x_{n_k}) with $\lim_{k\to\infty} x_{n_k} = a$, then (x_n) converges to a.*

Proof. Let $\varepsilon > 0$. Since (x_n) is Cauchy, there exists n_0 such that

$$m, n > n_0 \implies |x_n - x_m| < \frac{\varepsilon}{2}.$$

Since (x_{n_k}) converges to a, there exists k_0 such that

$$n_k > n_{k_0} \implies |x_{n_k} - a| < \frac{\varepsilon}{2}.$$

Choose $n_k > n_0$ satisfying this. Then for all $n > n_0$,

$$|x_n - a| \leq |x_n - x_{n_k}| + |x_{n_k} - a| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $\lim x_n = a$. □

By Lemma 3.44 and the Bolzano–Weierstrass Theorem, every Cauchy sequence has a convergent subsequence. Therefore, by Lemma 3.45, the sequence converges. This completes the proof of Theorem 3.43. □

The converse of Theorem 3.43 is also true:

Theorem 3.46. *Every convergent sequence is a Cauchy sequence.*

Proof. Suppose $a := \lim x_n$. Given $\varepsilon > 0$, there exist $n_0, n_1 \in \mathbb{N}$ such that

$$n > n_0 \implies |x_n - a| < \frac{\varepsilon}{2}, \quad m > n_1 \implies |x_m - a| < \frac{\varepsilon}{2}.$$

Then, for $m, n > \max\{n_0, n_1\}$,

$$|x_n - x_m| \leq |x_n - a| + |x_m - a| < \varepsilon.$$

Hence, x_n is Cauchy. □

Corollary 3.47. *A sequence x_n of real numbers is a Cauchy sequence if and only if it converges.*

A divergent sequence x_n *converges to infinity*, denoted by $\lim x_n = +\infty$, if for any $M > 0$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \implies x_n > M$. Similarly, x_n *converges to negative infinity*, denoted by $\lim x_n = -\infty$, if for any $M > 0$, there exists $n_0 \in \mathbb{N}$ such that $n > n_0 \implies x_n < -M$.

Example 3.48. The sequence $x_n = n$ converges to infinity. Given any $M > 0$, take $n_0 > M$. Then $x_n = n > M$ for all $n > n_0$. On the other hand, $x_n = (-1)^n n$ is divergent and does not converge to $+\infty$ or $-\infty$, since it is unbounded in both directions.

The following Theorem is similar to Theorem 3.25. The proof will be omitted.

Theorem 3.49. *(Arithmetic operations with infinite limits)*

1. *If* $\lim x_n = +\infty$ *and* y_n *is bounded from below, then* $\lim(x_n + y_n) = +\infty$ *and* $\lim(x_n \cdot y_n) = +\infty$.
2. *If* $x_n > 0$*, then* $\lim x_n = 0$ *if and only if* $\lim \frac{1}{x_n} = +\infty$.
3. *Let* $x_n, y_n > 0$ *be positive sequences. Then:*
   (a) *If* x_n *is bounded from below and* $\lim y_n = 0$*, then* $\lim \frac{x_n}{y_n} = +\infty$.
   (b) *If* x_n *is bounded and* $\lim y_n = +\infty$*, then* $\lim \frac{x_n}{y_n} = 0$.

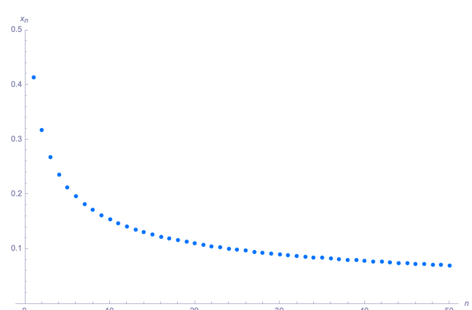


Fig. 3.6: $x_n = \sqrt{n+1} - \sqrt{n}$

Example 3.50. Let $x_n = \sqrt{n+1}$ and $y_n = -\sqrt{n}$. Then $\lim x_n = +\infty$, $\lim y_n = -\infty$. We have

$$\lim(x_n + y_n) = \lim \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \lim \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

However, it is **not true in general** that $\lim(x_n + y_n) = \lim x_n + \lim y_n$ if both limits are infinite. For example, $x_n = n^2$ and $y_n = -n$ satisfy $\lim x_n = +\infty$, $\lim y_n = -\infty$, but $\lim(x_n + y_n) = +\infty$.

Example 3.51. Let $x_n = [2 + (-1)^n]n$ and $y_n = n$. Then $\lim x_n = \lim y_n = +\infty$, but $\lim \frac{x_n}{y_n} = \lim[2 + (-1)^n]$ does not exist. Hence, it is not true in general that $\lim \frac{x_n}{y_n} = 1$ if both limits are $+\infty$.

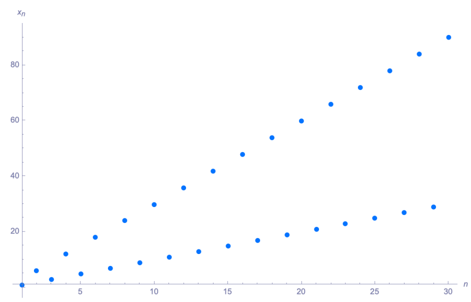


Fig. 3.7: $x_n = [2 + (-1)^n]n$

Example 3.52. Let $a > 1$. Then $\lim \frac{a^n}{n} = +\infty$. Indeed, write $a = 1 + s$ with $s > 0$. Then $a^n = (1 + s)^n \geq 1 + ns + \frac{n(n-1)}{2}s^2$ for $n \geq 2$, and

$$\lim \frac{1 + ns + \frac{n(n-1)}{2}s^2}{n} = +\infty.$$

By induction, one can show that for any $m \in \mathbb{N}$, $\lim \frac{a^n}{n^m} = +\infty$.

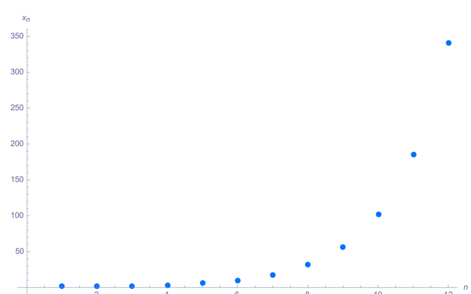


Fig. 3.8: $x_n = \sqrt{n+1} - \sqrt{n}$

Example 3.53. Let $a > 0$. Then $\lim \frac{n!}{a^n} = +\infty$. Indeed, pick $n_0 \in \mathbb{N}$ such that $\frac{n_0}{a} > 2$. Then

$$\frac{n!}{a^n} = \frac{n(n-1)\cdots(n_0+1)\,n_0!}{a^{n_0}\underbrace{a\cdots a}_{n-n_0}} > \frac{n_0!}{a^{n_0}}2^{\,n-n_0},$$

and it follows that $\lim \frac{n!}{a^n} = +\infty$.

3.4 Series

Given a sequence of real numbers (x_n), the purpose of this section is to give meaning to expressions of the form

$$x_1 + x_2 + x_3 + \cdots,$$

that is, the formal sum of all the terms of the sequence (x_n).

A natural way to do this is to define the sequence of *partial sums*

$$s_n := x_1 + x_2 + \cdots + x_n,$$

and then set

$$\sum_{n=1}^{\infty} x_n := \lim_{n\to\infty} s_n,$$

whenever this limit exists.

It is common practice to write simply $\sum x_n$ instead of $\sum_{n=1}^{\infty} x_n$, and to call x_n the *general term* of the series. We shall adopt these conventions in this book.

Since the definition of $\sum x_n$ involves a limit, the series may or may not converge. If

$$\sum x_n = L \in \mathbb{R},$$

we say that the series $\sum x_n$ *converges* to L. Otherwise, we say that $\sum x_n$ *diverges*.

Theorem 3.54. *If the series $\sum x_n$ converges, then $\lim x_n = 0$.*

Proof. Indeed, since $x_n = s_n - s_{n-1}$, we obtain

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}(s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = 0.$$

□

The converse of the theorem above is not true, as the following counterexample shows.

Example 3.55. (Harmonic series) Consider the series $\sum \frac{1}{n}$. We clearly have $\lim \frac{1}{n} = 0$. However, we claim that $\sum \frac{1}{n}$ diverges.

To prove that $\lim s_n$ diverges, it suffices to exhibit a divergent subsequence of s_n. Consider the subsequence s_{2^n}:

$$\begin{aligned}
s_{2^n} &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} \\
&= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots \\
&> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + \cdots + \frac{2^{n-1}}{2^n} \\
&= 1 + n \cdot \frac{1}{2}.
\end{aligned}$$

Hence, $s_{2^n} > 1 + \frac{n}{2}$, and therefore $\lim s_{2^n} = +\infty$. Thus the harmonic series diverges.

Example 3.56. (Geometric series) Consider the series $\sum a^n$ with $a \in \mathbb{R}$.

If $|a| \geq 1$, then the general term $x_n = a^n$ does not satisfy $\lim x_n = 0$, so the series diverges.

If $|a| < 1$, then the series converges. Indeed, by induction one shows

$$s_n = \frac{1 - a^{n+1}}{1 - a}.$$

Taking the limit as $n \to \infty$ gives

$$\sum_{n=0}^{\infty} a^n = \lim_{n\to\infty} s_n = \frac{1}{1-a}, \qquad (|a| < 1).$$

Theorem 3.57. *Let a_n and b_n be real sequences and consider the series $\sum a_n$ and $\sum b_n$. Then:*

1. *If $\sum a_n$ and $\sum b_n$ converge, then $\sum(a_n + b_n)$ converges and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n.$$

2. *For any $c \in \mathbb{R}$, if $\sum a_n$ converges then $\sum(c\, a_n)$ converges and*

$$\sum_{n=1}^{\infty} c\, a_n = c \sum_{n=1}^{\infty} a_n.$$

Proof. The proof follows directly from the properties of limits as shown below.

(1) Let $S_n = \sum_{k=1}^{n} a_k$ and $T_n = \sum_{k=1}^{n} b_k$ be the partial sums of the two series, and suppose $S_n \to S$ and $T_n \to T$. The nth partial sum of $\sum(a_n + b_n)$ is $S_n + T_n$, hence

$$\lim_{n\to\infty}(S_n + T_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} T_n = S + T,$$

so $\sum(a_n + b_n)$ converges to $S + T$.

(2) If $S_n = \sum_{k=1}^{n} a_k \to S$, then the nth partial sum of $\sum c\, a_n$ is cS_n, and

$$\lim_{n\to\infty} cS_n = c \lim_{n\to\infty} S_n = cS,$$

so $\sum c\, a_n$ converges to cS. □

Example 3.58. (Telescoping series) Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$. Since

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1},$$

the nth partial sum is

$$s_n = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n+1}.$$

Hence $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} s_n = 1.$

Example 3.59. The series $\displaystyle\sum_{n=1}^{\infty} (-1)^n$ is divergent. Indeed, the general term $(-1)^n$ does not tend to 0 (it has two accumulation points, 1 and -1), and by Theorem 3.54 a necessary condition for convergence is that the terms tend to 0. Therefore the series cannot converge.

Theorem 3.60. *Let a_n be a sequence of nonnegative real numbers, $a_n \geq 0$ for all n. The series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums s_n is bounded.*

Proof. Let $s_n = \sum_{k=1}^{n} a_k$ be the partial sums. Observe that s_n is monotone nondecreasing because each $a_n \geq 0$ implies $s_{n+1} - s_n = a_{n+1} \geq 0$.

If $\sum a_n$ converges, then by definition s_n has a finite limit and is therefore bounded.

Conversely, if (s_n) is bounded and monotone nondecreasing, then by the Monotone Convergence Theorem it converges. Thus $\sum a_n = \lim_{n\to\infty} s_n$ exists and is finite. □

Corollary 3.61. *(Comparison principle) Suppose $\sum a_n$ and $\sum b_n$ are series of nonnegative real numbers, i.e. $a_n, b_n \geq 0$. If there are $c \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $a_n \leq c\, b_n$ for $n > n_0$, then if $\sum b_n$ converges, $\sum a_n$ converges. Moreover, if $\sum a_n$ diverges then $\sum b_n$ diverges.*

Example 3.62. If $r > 1$, the series $\sum \frac{1}{n^r}$ converges. Indeed, the general term of this series is positive, so the partial sums s_n are increasing, hence it's enough to prove that a subsequence of s_n is bounded. We claim s_{2^n-1} is bounded. We have:

$$\begin{aligned} s_{2^n-1} &= 1 + \frac{1}{2^r} + \ldots + \frac{1}{(2^n-1)^r} \\ &= 1 + \left(\frac{1}{2^r} + \frac{1}{3^r}\right) + \left(\frac{1}{4^r} + \frac{1}{5^r} + \frac{1}{6^r} + \frac{1}{7^r}\right) + \ldots + \frac{1}{(2^n-1)^r} \\ &< 1 + \frac{2}{2^r} + \frac{4}{4^r} + \frac{8}{8^r} + \ldots + \frac{2^{n-1}}{2^{(n-1)r}} \\ &= \sum_{j=0}^{n-1} \left(\frac{2}{2^r}\right)^j \end{aligned}$$

On the other hand, the geometric series $\sum_{j=0}^{\infty} \left(\frac{2}{2^r}\right)^j$ converges since $\frac{2}{2^r} < 1$. We conclude that s_{2^n-1} is bounded and the claim follows.

Example 3.63. (p-series) Let $p > 1$. We claim that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges. Let s_k denotes its partial sum, i.e. $s_k = \sum_{n=1}^{k} \frac{1}{n^p}$. Since s_k is increasing, in order to prove its convergence, it's enough to prove that it is bounded as well. Take $k = 2^m - 1$, then

$$\begin{aligned} s_k &= 1 + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \ldots + \frac{1}{(2^m-1)^p} \\ &< 1 + \frac{2}{2^p} + \frac{4}{4^p} + \ldots + \frac{2^{m-1}}{2^{(m-1)p}} = \sum_{i=0}^{m-1} \left(\frac{2}{2^p}\right)^i. \end{aligned}$$

Since $p > 1$, it follows that $\frac{2}{2^p} < 1$ and the geometric series $\sum_{i=0}^{m-1} \left(\frac{2}{2^p}\right)^i$ converges. Hence, s_k is bounded, as desired.

Corollary 3.64. *(Cauchy's criteria) The series $\sum a_n$ is convergent if and only if given $\epsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $|a_{n+1} + \ldots + a_{n+p}| < \epsilon$ for $n > n_0$.*

Proof. Notice that s_n converges if and only if it is a Cauchy sequence (see Corollary 3.47). □

A series $\sum a_n$ is **absolutely convergent** if $\sum |a_n|$ is convergent. A series with all of its terms positive (or negative) is convergent if and only if is absolutely convergent. Hence, in this case the two notion coincide. Here's a classical counterexample that shows that they don't coincide in general:

Example 3.65. Consider the series $\sum \frac{(-1)^n}{n}$. We already know that $\sum \frac{1}{n}$ diverges, however we claim that $\sum \frac{(-1)^n}{n}$ converges. Indeed, notice that the subsequence s_{2n} satisfies

$$s_2 < s_4 < s_6 < \ldots < s_{2n},$$

and is a Cauchy sequence, hence convergent. Whereas s_{2n-1} satisfies

$$s_1 > s_3 > s_5 > \ldots > s_{2n-1},$$

so it's bounded and monotone, hence convergent as well. Set $a := \lim s_{2n}$, $b := \lim s_{2n-1}$, then since $s_{2n} - s_{2n-1} = \frac{1}{2n} \to 0$, we necessarily have $a = b$. We conclude that s_n has only one accumulation point, hence converges. (We will see later that $a = b = \log 2$)

A series $\sum a_n$ is **conditionally convergent** if $\sum a_n$ is convergent but $\sum |a_n|$ is divergent. The example above shows that $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 3.66. *Every absolutely convergent series $\sum a_n$ is convergent.*

Proof. By hypothesis, $\sum a_n$ is Cauchy, so we can find $n_0 \in \mathbb{N}$ such that $n > n_0, \forall p \in \mathbb{N} \Rightarrow |a_{n+1}| + \ldots + |a_{n+p}| < \epsilon$. In particular, $|a_{n+1} + \ldots + a_{n+p}| < |a_{n+1}| + \ldots + |a_{n+p}| < \epsilon$, the conclusion follows from Cauchy's criteria (Corollary 3.64). □

Corollary 3.67. *Let $\sum b_n$ a convergent series with $b_n \geq 0$. If there are $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $n > n_0 \Rightarrow |a_n| \leq c\, b_n$ then the series $\sum a_n$ is absolutely convergent.*

Corollary 3.68. *(The root test) If there are $n_0 \in \mathbb{N}$ and $c \in \mathbb{R}$ such that $n > n_0 \Rightarrow \sqrt[n]{|a_n|} \leq c < 1$, then the series $\sum a_n$ is absolutely convergent. In other words, if $\limsup \sqrt[n]{|a_n|} < 1$ then $\sum a_n$ is absolutely convergent. On the other hand, if $\limsup \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ diverges.*

Proof. In this case, we can compare $\sum |a_n|$ with $\sum c^n$, the latter (absolutely) converges since it's a geometric series with $0 < c < 1$. If $\sqrt[n]{|a_n|} > 1$ for n sufficiently large, then $\lim a_n \neq 0$. □

Corollary 3.69. *(The root test – second version) If $\lim \sqrt[n]{|a_n|} < 1$, then the series $\sum a_n$ is absolutely convergent. If $\lim \sqrt[n]{|a_n|} > 1$, then the series $\sum a_n$ is divergent.*

Example 3.70. Let $a \in \mathbb{R}$ and consider the series $\sum na^n$. Notice that $\lim \sqrt[n]{n\,|a|^n} = \lim \sqrt[n]{n} \lim |a| = |a|$. Hence, if $|a| < 1$ the series $\sum na^n$ is absolutely convergent and if $|a| > 1$ it diverges. If $|a| = 1$ the series also diverges, since $\lim na^n \neq 0$ in this case.

Theorem 3.71. *(The ratio test) Let $\sum a_n$ and $\sum b_n$ be series of real numbers such that $a_n \neq 0, b_n > 0, \forall n \in \mathbb{N}$ and $\sum b_n$ convergent. If there is $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \leq \left|\frac{b_{n+1}}{b_n}\right|$, then $\sum a_n$ is absolutely convergent.*

Proof. Consider the inequalities:

$$\left|\frac{a_{n_0+2}}{a_{n_0+1}}\right| \leq \left|\frac{b_{n_0+2}}{b_{n_0+1}}\right|$$
$$\left|\frac{a_{n_0+3}}{a_{n_0+2}}\right| \leq \left|\frac{b_{n_0+3}}{b_{n_0+2}}\right|$$
$$\dots$$
$$\left|\frac{a_n}{a_{n-1}}\right| \leq \left|\frac{b_n}{b_{n-1}}\right|$$

Multiplying them together, we have:

$$\left|\frac{a_n}{a_{n_0+1}}\right| \leq \left|\frac{b_n}{b_{n_0+1}}\right|$$

Hence, $|a_n| \leq c\, b_n$ and the result follows by the comparison principle. □

Corollary 3.72. *(The ratio test – second version) If* $\limsup\left|\frac{a_{n+1}}{a_n}\right| < 1$*, then the series* $\sum a_n$ *is absolutely convergent. If* $\limsup\left|\frac{a_{n+1}}{a_n}\right| > 1$*, then the series* $\sum a_n$ *is divergent.*

Proof. For the convergence, take $b_n = (\limsup\left|\frac{a_{n+1}}{a_n}\right|)^n$ in Theorem 3.71. If $\limsup\left|\frac{a_{n+1}}{a_n}\right| > 1$ then $\lim a_n \neq 0$. □

Corollary 3.73. *(The ratio test – third version) If* $\lim\left|\frac{a_{n+1}}{a_n}\right| < 1$ *then* $\sum a_n$ *is absolutely convergent, if* $\lim\left|\frac{a_{n+1}}{a_n}\right| > 1$ *then* $\sum a_n$ *diverges.*

Example 3.74. Fix $x \in \mathbb{R}$ and consider the series $\sum \frac{x^n}{n!}$, then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0$ regardless of x, and the series is absolutely convergent. We will see later that this series coincides with e^x.

Theorem 3.75. *(Root test is stronger than the ratio test) For any bounded sequence* a_n *of positive numbers we have*

$$\liminf \frac{a_{n+1}}{a_n} \leq \liminf \sqrt[n]{a_n} \leq \limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n},$$

In particular, if $\lim \frac{a_{n+1}}{a_n} = c$ *then* $\lim \sqrt[n]{a_n} = c$.

Proof. It's enough to prove that $\limsup \sqrt[n]{a_n} \leq \limsup \frac{a_{n+1}}{a_n}$, the first inequality can be proven *mutatis mutandis*. We argue by contradiction, suppose there is a $k \in \mathbb{R}$ such that

$$\limsup \sqrt[n]{a_n} > k > \limsup \frac{a_{n+1}}{a_n}$$

Proceeding as in the proof of Theorem 3.71, we can find $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow a_n < c\,k^n$, which implies that $\sqrt[n]{a_n} < c^{\frac{1}{n}}\,k$ and hence:

$$\limsup \sqrt[n]{a_n} \le k$$

a contradiction. □

Example 3.76. A nice application of the Theorem above is the computation of $\lim \frac{n}{\sqrt[n]{n!}}$. Set $x_n = \frac{n}{\sqrt[n]{n!}}$ and $y_n = \frac{n^n}{n!}$, then $x_n = \sqrt[n]{y_n}$. On the other hand, $\frac{y_{n+1}}{y_n} = (1 + \frac{1}{n})^n$, hence $\lim \frac{y_{n+1}}{y_n} = e$, and it follows that $\lim \frac{n}{\sqrt[n]{n!}} = e$.

Example 3.77. Given two distinct numbers $a, b \in \mathbb{R}$, consider the sequence $x_n = \{a, ab, a^2b, a^2b^2, a^3b^2, \ldots\}$, then the ratio $\frac{x_{n+1}}{x_n} = b$ if n is odd, and $\frac{x_{n+1}}{x_n} = a$ if n is even, hence the sequence $\frac{x_{n+1}}{x_n}$ doesn't converge and $\lim \frac{x_{n+1}}{x_n}$ doesn't exist. On the other hand, we have $\lim \sqrt[n]{x_n} = \sqrt{ab}$. This demonstrates that in the Theorem above the inequalities can be strict.

Theorem 3.78. *(Dirichlet) Let b_n be a nonincreasing sequence of positive numbers with $\lim b_n = 0$, and $\sum a_n$ be a series such that the partial sum s_n is a bounded sequence. Then the series $\sum a_n b_n$ converges.*

Proof. Notice that

$$\begin{aligned} a_1b_1 + a_2b_2 + \ldots + a_nb_n &= a_1(b_1 - b_2) + (a_1 + a_2)(b_2 - b_3) + \\ &\quad + (a_1 + a_2 + a_3)(b_3 - b_4) + \ldots + (a_1 + \ldots + a_n)b_n \\ &= \sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i) + s_nb_n \end{aligned}$$

Since s_n is bounded, say $|s_n| \le k$ and $b_n \to 0$, we have $\lim s_nb_n = 0$. Moreover, $|\sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i)| \le k|\sum_{i=2}^{n}(b_{i-1} - b_i)| = k(b_1 - b_n)$. So $\sum_{i=2}^{n} s_{i-1}(b_{i-1} - b_i)$ converges, and therefore, by comparison, $\sum a_nb_n$ converges as well. □

We can weaken the hypothesis $\lim b_n = 0$ if we take $\sum a_n$ convergent. Indeed, if $\lim b_n = c$ just take $b_n^* := b_n - c$ and use this new sequence instead. We conclude:

Corollary 3.79. *(Abel) If $\sum a_n$ is convergent and b_n is a nonincreasing sequence of positive numbers then $\sum a_nb_n$ converges.*

Corollary 3.80. *(Leibniz) Let b_n be a nonincreasing sequence of positive numbers with $\lim b_n = 0$. Then the series $\sum(-1)^n b_n$ converges.*

Proof. In this case, $a_n = (-1)^n$ has bounded partial sum, namely $|s_n| \le 1$, and the result follows directly from Theorem 3.78. □

Example 3.81. Some periodic real valued functions can be written as a linear combination of $\sum \cos(nx)$ and $\sum \sin(nx)$. The properties of such functions and generalizations are addressed in area of mathematics called **Fourier Analysis**. E. Stein's book on the subject is a wonderful first-read of the topic.

Take the example of $f(x) = \sum \frac{\cos(nx)}{n}$, we claim that if $x \neq 2\pi k$, $k \in \mathbb{Z}$ then $f(x)$ is well-defined, i.e. $\sum \frac{\cos(nx)}{n}$ converges. Indeed, let $a_n = \cos(nx)$ and $b_n = \frac{1}{n}$, then b_n is decreasing, so by Theorem 3.78, it's enough to prove that the partial sums s_n of $\sum a_n$ are bounded. In other words, we need to show that

$$s_n = \cos(x) + \cos(2x) + \cos(3x) + \ldots + \cos(nx)$$

is bounded. Recall, that $e^{ix} = \cos(x) + i\sin(x)$. Therefore:

$$1 + s_n = \mathrm{Re}[1 + e^{ix} + e^{2ix} + e^{3ix} + \ldots + e^{nix}]$$
$$1 + s_n = \mathrm{Re}[\frac{1 - e^{(n+1)ix}}{1 - e^{ix}}]$$
$$1 + s_n \leq \frac{2}{|1 - e^{ix}|}$$

It follows that s_n is bounded and we conclude that $\sum \frac{\cos(nx)}{n}$ converges if $x \neq 2\pi k$.

Given a series $\sum a_n$, we define the *positive part* of $\sum a_n$ as the series $\sum p_n$, where $p_n = a_n$ if $a_n > 0$, and $p_n = 0$ if $a_n \leq 0$. Similarly, the *negative part* of $\sum a_n$ as the series $\sum q_n$, where $q_n = -a_n$ if $a_n < 0$, and $q_n = 0$ if $a_n \geq 0$. It follows immediately from the definition that $p_n, q_n \geq 0$ and $a_n = p_n - q_n, |a_n| = p_n + q_n \; \forall n \in \mathbb{N}$.

Proposition 3.82. *The series $\sum a_n$ is absolutely convergent if and only if $\sum p_n$ and $\sum q_n$ converge.*

Proof. Notice that $p_n \leq |a_n|$ and $q_n \leq |a_n|$, hence if $\sum |a_n|$ converge then by comparison $\sum p_n$ and $\sum q_n$ also converge. The converse is obvious. □

Example 3.83. If $\sum a_n$ is not absolutely convergent, then the proposition is false. Take the example of $\sum \frac{(-1)^n}{n}$. In this case, $\sum p_n = \sum \frac{1}{2n}$ and $\sum q_n = \sum \frac{1}{2n-1}$, and both diverge.

Proposition 3.84. *If $\sum a_n$ is conditionally convergent then $\sum p_n$ and $\sum q_n$ diverge.*

Proof. Suppose not, say $\sum q_n$ converge. Then $\sum |a_n| = \sum p_n + \sum q_n = \sum a_n + 2\sum q_n$ also converges, a contradiction. □

Let $f : \mathbb{N} \to \mathbb{N}$ be a bijection and $\sum a_n$ be a series of real numbers. Set $b_n = a_{f(n)}$. We say $\sum a_n$ is **commutatively convergent** if $\sum a_n = \sum b_n$ for every bijection $f : \mathbb{N} \to \mathbb{N}$. We will show below that the notion of commutative convergence coincides with absolute convergence.

Theorem 3.85. *A series $\sum a_n$ is absolutely convergent if and only if is commutatively convergent.*

Proof. Suppose $\sum a_n$ absolutely convergent, and let $b_n = a_{f(n)}$ for some bijection $f : \mathbb{N} \to \mathbb{N}$. It's enough to assume that $a_n \geq 0$, otherwise just use the fact that $a_n = p_n - q_n$, for $p_n, q_n \geq 0$, and apply the result for p_n and q_n. Now, fix $n \in \mathbb{N}$ and let $s_n = \sum_{i=1}^{n} a_i$ denote the partial sum of $\sum a_n$, and $t_n = \sum_{i=1}^{n} b_i$, the partial sum of $\sum b_n$. If we set $m := \max\{f(x); 1 \leq x \leq n\}$, it follows that $t_n = \sum_{i=1}^{n} a_{f(i)} \leq \sum_{i=1}^{m} a_i = s_m$. We conclude that for each $n \in \mathbb{N}$ it's possible to find $m \in \mathbb{N}$ such that $t_n \leq s_m$, and similarly using $f^{-1}(y)$ instead of $f(x)$, given $m \in \mathbb{N}$ it's possible to find $n \in \mathbb{N}$, such that $s_m \leq t_n$, which implies $\lim s_n = \lim t_n$, hence $\sum a_n = \sum b_n$.

Conversely, we want to show that if $\sum a_n$ is commutatively convergent then it is absolutely convergent. We prove the contra-positive, that is, suppose $\sum a_n$ is not absolutely convergent then $\sum a_n$ is not commutatively convergent. Indeed, if $\sum a_n$ is divergent, just take $b_n = a_n$. Otherwise, $\sum a_n$ is conditionally convergent, say $\sum a_n = S \in \mathbb{R}$, and by proposition 3.84, both $\sum p_n$ and $\sum q_n$ diverge. Moreover, since $\lim a_n = 0$, we have $\lim p_n = \lim q_n = 0$. Take any number $c \neq S$, we will show that we can reorder a_n into b_n in such a way that $\sum b_n = c$, hence $\sum a_n$ can't be commutatively convergent. Let n_1 be the smallest natural such that

$$p_1 + p_2 + \ldots + p_{n_1} > c,$$

and $n_2 > n_1$, be smallest number such that

$$p_1 + \ldots + p_{n_1} - q_1 - q_2 - \ldots - q_{n_2} < c.$$

Proceeding by induction, we obtain a new series $\sum b_n$, such that the partial sums t_n approach c. Indeed, for odd i we have $t_{n_i} - c \leq p_{n_i}$, be definition of n_i, and similarly, $c - t_{n_{i+1}} \leq q_{n_{i+1}}$. Since $\lim p_n = \lim q_n = 0$, we have $\lim t_{n_i} = c$. A similar argument holds for i even. □

Exercises

1. If $\lim x_n = a$, show that $\lim |x_n| = |a|$. Show that the converse can be false by giving a counter example.
2. Suppose $\lim x_n = 0$. Let $y_n = \min\{|x_1|, |x_2|, \ldots, |x_n|\}$. Show that $\lim y_n = 0$.
3. If $\lim x_{2n} = a$ and $\lim x_{2n-1} = a$, show that $\lim x_n = a$.
4. Given an example of a sequence x_n and a infinite decomposition of $\mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_k \cup \ldots$, such that for every $k \in \mathbb{N}$, the subsequence $(x_n)_{n\in\mathbb{N}_k}$ has limit $a \in \mathbb{R}$ but $\lim x_n \neq a$.
5. If $\lim x_n = a$ and $\lim(x_n - y_n) = 0$, show that $\lim y_n = a$.
6. Show that $(1 - \frac{1}{n})^n$ is increasing. *Hint: Use the inequality of arithmetic and geometric means involving the* $n + 1$ *numbers* $1 - \frac{1}{n}, \ldots, 1 - \frac{1}{n}, 1$.
7. Let $x_n = (1 + \frac{1}{n})^n$, $y_n = (1 - \frac{1}{n+1})^{n+1}$. Show that $\lim x_n y_n = 1$ and conclude that $\lim(1 - \frac{1}{n})^n = e^{-1}$.
8. Let $a \geq 0, b \geq 0$. Show that $\lim \sqrt[n]{a_n + b_n} = \max\{a, b\}$
9. Let x_n be a bounded sequence. If $\lim a_n = a$ and a_n is an accumulation point of x_n, then a is an accumulation point of x_n.
10. Let x_n, y_n be bounded sequences. Set

$$a = \liminf x_n, A = \limsup x_n, b = \liminf y_n, B = \limsup y_n$$

Show that:

a) $\limsup(x_n + y_n) \leq A + B$ and $\liminf(x_n + y_n) \geq a + b$;
b) $\limsup -x_n = -a$ and $\liminf -x_n = -A$;
c) If $x_n \geq 0, y_n \geq 0$, then $\limsup(x_n \cdot y_n) \leq A \cdot B$ and $\liminf(x_n \cdot y_n) \geq a \cdot b$.

11. For each $n \in \mathbb{N}$, let $0 \leq t_n \leq 1$. If $\lim x_n = \lim y_n = a$, show that

$$\lim[t_n x_n + (1 - t_n) y_n] = a$$

12. Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$. Show that x_n is bounded and find $\lim x_n$.
13. Show that x_n doesn't have a convergent subsequence if and only if $\lim |x_n| = +\infty$.
14. Let $y_n > 0$ for every $n \in \mathbb{N}$, such that $\sum y_n = +\infty$. If x_n is a sequence such that $\lim \frac{x_n}{y_n} = a$, show that $\lim \frac{x_1 + \ldots + x_n}{y_1 + \ldots + y_n} = a$.
15. Let y_n be an increasing sequence and $\lim y_n = +\infty$. Show that

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = a \Rightarrow \lim \frac{x_n}{y_n} = a$$

16. Show that

$$\lim \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \frac{1}{p+1}$$

17. Show that for every $n \in \mathbb{N}$, $0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) < \frac{1}{n!n}$. Conclude that $e \notin \mathbb{Q}$.
18. Show that $\lim \frac{1}{2} \sqrt[n]{(n+1)(n+2)\ldots 2n} = \frac{4}{e}$.
19. Suppose the sequence x_n satisfies $n! = n^n e^{-n} x_n$. Show that $\lim \sqrt[n]{x_n} = 1$.
20. Let $\sum a_n$ and $\sum b_n$ be series with positive elements. Show that if $\sum b_n = +\infty$ and $\exists n_0 \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$ for $n > n_0$, then $\sum a_n = +\infty$.
21. Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree 2 or more. Show that the series $\sum \frac{1}{p(n)}$ converges.
22. If $|x| < 1$ show that $\lim\limits_{n\to\infty} \binom{m}{n} x^n = 0$ for every $m \in \mathbb{R}$, where $\binom{m}{n} := \frac{m(m-1)\ldots(m-n+1)}{n!}$.
23. Let $a \in \mathbb{R}$. Show that the series $\sum\limits_{n=0}^{\infty} \frac{a^2}{(1+a^2)^n}$ converges and find its sum.
24. Show that for every fixed $p \in \mathbb{R}$, the series $\sum \frac{1}{n(n+1)\ldots(n+p)}$ converges.
25. If $\sum a_n$ converges and $a_n > 0$ then $\sum a_n^2$ and $\frac{a_n}{1+a_n}$ also converge.
26. If $\sum a_n^2$ converges then $\frac{a_n}{n}$ also converges.
27. If a_n is decreasing and $\sum a_n$ converges then $\lim a_n \cdot n = 0$.
28. If a_n is nonincreasing with $\lim a_n = 0$, show that $\sum a_n$ converges if and only if $\sum 2^n \cdot a_{2^n}$ converges.
29. Show that the set of accumulation points of the sequence $x_n = \cos n$ is the closed interval $[-1, 1]$.
30. Let $a_1 \geq a_2 \geq \ldots \geq 0$ and $s_n = a_1 - a_2 + \ldots + (-1)^{n-1} a_n$. Show that s_n is bounded and

$$\limsup s_n - \liminf s_n = \lim a_n$$

Chapter 4
Topology of $\mathbb{R}$

4.1 Open sets

Let $X \subseteq \mathbb{R}$. A point $p \in X$ is called *an interior point* if there is an open interval (a, b), also called a *neighborhood*, such that $p \in (a, b) \subseteq X$. In other words, p is an interior point if all points sufficiently close to p remain in X.

It's easy to see that $p \in X$ is an interior point if and only if $\exists \epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subseteq X$. Equivalently, p is an interior point if and only if$\exists \epsilon > 0$ such that $|x - p| < \epsilon \Rightarrow x \in X$.

The set of all interior points of X, denoted by $\text{int}(X)$ (also by X°), is called *the interior of* X. Notice that by definition, we necessarily have $\text{int}(X) \subseteq X$.

A set $X \subseteq \mathbb{R}$ is **open** if $X = \text{int}(X)$. That is to say, every point of X is an interior point.

Example 4.1. By definition if X has an interior point then it contains an open interval, in particular it is an infinite set. Hence, if $X = \{x_1, \ldots, x_n\}$ is finite then it has no interior points. Moreover, if $\text{int}(X) \neq \emptyset$ then X is uncountable since it contains an interval. Therefore,

$$\text{int}(\mathbb{N}) = \text{int}(\mathbb{Z}) = \text{int}(\mathbb{Q}) = \emptyset,$$

and they can't be open sets. Similarly, since $\mathbb{Q}$ is dense, any open interval containing an irrational point also contains a rational point, hence

$$\text{int}(\mathbb{R} - \mathbb{Q}) = \emptyset,$$

and it's not open as well.

Example 4.2. The open interval (a, b) is open. Indeed, any $x \in (a, b)$ is an interior point because (a, b) itself contains x. On the other hand, the closed interval $[a, b]$ is not open because $\text{int}([a, b]) = (a, b) \neq [a, b]$. Indeed, any open interval containing the endpoints necessarily contain points outside $[a, b]$, so the endpoints can't be interior points. Similarly, if $X = [a, b)$ or $X = (a, b]$ then $\text{int}(X) = (a, b)$

Example 4.3. The empty set $\emptyset$ is open since its interior is also empty, i.e. $\text{int}(\emptyset) = \emptyset$.

Example 4.4. The union of two open intervals $X = (a, b) \cup (c, d)$ is open. Indeed, any interior point of X has to be an interior point of (a, b) or (c, d).

Theorem 4.5. *a) If $A, B \subseteq \mathbb{R}$ are open then $A \cap B$ is open*

b) Given an arbitrary set L. If $\{A_i\}_{i \in L}$ is a family of open sets, then $\bigcup\limits_{i \in L} A_i$ is open.

Proof. a) Let $x \in A \cap B$, then we can find $a, b, c, d \in \mathbb{R}$ such that $x \in (a, b) \subseteq A$ and $x \in (c, d) \subseteq B$. Let $m := \max\{a, c\}$ and $M := \min\{b, d\}$, then $x \in (m, M) \subseteq A \cap B$.

b) Let $x \in \bigcup\limits_{i \in L} A_i$, then there is at least one $i_0 \in L$ such that $x \in A_{i_0}$. Since A_{i_0} is open by definition, we can find a neighborhood $(a, b) \ni x$ such that $(a, b) \subseteq A_{i_0} \subseteq \bigcup\limits_{i \in L} A_i$. We conclude that every point is an interior point.

□

Corollary 4.6. *Every open set $X \subseteq \mathbb{R}$ is a union of open intervals.*

Proof. For each $x \in X$, take an open interval $I_x \ni x$ such that $I_x \subseteq X$. Then $X = \bigcup\limits_{x \in X} I_x$. □

Corollary 4.7. *If $A_1, A_2, \ldots, A_n$ are open sets then $A_1 \cap A_2 \cap \ldots \cap A_n$ is an open set.*

The corollary above is false for countably infinite intersections, take for example the open intervals $A_n = (-\frac{1}{n}, \frac{1}{n})$. Then $\bigcap\limits_{i=1}^{\infty} A_i = \{0\}$, which is not open (since it's finite).

Example 4.8. Let $a \in \mathbb{R}$, then the set $X = \mathbb{R} - \{a\}$ is open. Indeed, set $A = (-\infty, a)$ and $B = (a, +\infty)$. Then both A and B are open and $X = A \cup B$, hence X is open. More generally, we can use induction to show that $\mathbb{R} - \{a_1, \ldots, a_n\}$ is open.

Before proving the next theorem, we need the following lemma:

Lemma 4.9. *Let $\{I_j\}_{j \in L}$ be a family of open intervals containing a point $x \in \mathbb{R}$. Then $I = \bigcup\limits_{j \in L} I_j$ is itself an open interval.*

Proof. Suppose $I_j = (a_j, b_j)$. By hypothesis,

$$a_j < x < b_j, \ \forall j \in L.$$

Set $a := \inf a_j$ and $b := \sup b_j$ (Notice that it's possible that $a = -\infty, b = +\infty$.) We claim that $I = (a, b)$. The inclusion $I \subseteq (a, b)$ is clear. Conversely, let $y \in (a, b)$. Then by definition of supremum and infimum, we can find a_j and

b_k such that $a_j < y < b_k$, if $y < b_j$ then $y \in I_j$. Otherwise, $y \geq b_j$, and $a_j < b_j \leq y$, which implies that $a_k < y < b_k$, and $y \in I_k$. In conclusion, $(a, b) \subseteq I$, hence $I = (a, b)$. □

Theorem 4.10. *(Structure of open sets) Every open set $X \subseteq \mathbb{R}$ can be written uniquely as a countable union of pairwise disjoints open intervals, called the interval components of X.*

Proof. Given $x \in X$, let I_x be the union of all open intervals I_j contained in X such that $I_j \ni x$. By lemma 4.9, I_x is an open interval. We claim that either $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Indeed, if $I_x \cap I_y \neq \emptyset$ then $I_x \cap I_y$ itself is an interval containing, say x, hence $I_x \cap I_y \subseteq I_x$, and $I_y \subseteq I_x$. Similarly, $I_x \cap I_y \subseteq I_y \Rightarrow I_x \subseteq I_y$ and it follows that $I_x = I_y$.

Define $L = \{\overline{x} \in X; x \sim y \text{ if } I_x = I_y\}$, that is, L is constructed by identifying elements of X who have the same component. Then X is the union $X = \bigcup\limits_{\overline{x} \in L} I_x$ of pairwise disjoints open intervals. In order to prove that this union is countable we define a function that associates to each $\overline{x} \in L$ a random rational number $r(\overline{x}) \in \mathbb{Q}$ contained in I_x. Since $I_x \neq I_y \Rightarrow I_x \cap I_y = \emptyset \Rightarrow r(\overline{x}) \neq r(\overline{y})$, hence the function $r : L \longrightarrow \mathbb{Q}$ is injective and corollary **??** implies that L is countable.

We are left to prove uniqueness. Suppose $X = \bigcup\limits_{i=k}^{\infty} J_k$, where J_k are open intervals, say $J_k = (a_k, b_k)$, pairwise disjoints. We claim the endpoints of J_k are not in X. Indeed, if $a_k \in X$ then $\exists J_l$ such that $a_k \in (a_l, b_l)$, but then if we set $b := \min\{b_k, b_l\}$, we have $(a_k, b) \subseteq J_k \cap J_l$, a contradiction since $J_k \cap J_l = \emptyset$. Therefore, for each $x \in J_k$, J_k is the largest open interval containing x inside X, and we must have $J_k = I_x$. □

Corollary 4.11. *(Connectedness of intervals) Let $I \subseteq \mathbb{R}$ be an open interval. If $I = A \cup B$, where A and B are open and $A \cap B = \emptyset$, then either $A = I$ or $B = I$ ($B = \emptyset$ or $A = \emptyset$.)*

4.2 Closed sets

We say a point $a \in \mathbb{R}$ is *adherent (or closure point)* of the set $X \subseteq \mathbb{R}$ if it is limit of a sequence of points in X. Every point of X is adherent to itself, since any point $x \in X$ is the limit of the constant sequence $x_n = x$.

Example 4.12. Consider $X = (0, +\infty)$. Then $0 \notin X$ but 0 is an adherent point, since $0 = \lim x_n$, where $x_n = \frac{1}{n} \in X$.

Theorem 4.13. *A point $a \in \mathbb{R}$ is adherent of the set $X \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$, $(a - \epsilon, a + \epsilon) \cap X \neq \emptyset$.*

Proof. Suppose a is an adherent point, say $\lim x_n = a$, where $x_n \in X$. Given any $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $n > n_0 \Rightarrow x_n \in (a - \epsilon, a + \epsilon)$, in particular, $(a-\epsilon, a+\epsilon) \cap X \neq \emptyset$. Conversely, suppose $(a-\epsilon, a+\epsilon) \cap X \neq \emptyset$ for every $\epsilon > 0$. By choosing $\epsilon = \frac{1}{n}$ for each $n \in \mathbb{N}$, we are able to construct a sequence $x_n \in X$ such that $x_n \in (a - \frac{1}{n}, a + \frac{1}{n})$, and hence $\lim x_n = a$. □

Corollary 4.14. *A point $a \in \mathbb{R}$ is adherent of the set $X \subseteq \mathbb{R}$ if and only if every open interval $I \ni a$ we have $I \cap X \neq \emptyset$.*

Corollary 4.15. *Suppose $X \subseteq \mathbb{R}$ is bounded, then* $\sup X$ *and* $\inf X$ *are adherent points.*

The set of all adherent points of X, denoted by $\overline{X}$ is called the *closure* of X. A set $X \subseteq \mathbb{R}$ is **closed** if $X = \overline{X}$. In other words, a set X is closed if and only if it contains all of its adherent points.

Notice that a set $X \subseteq \mathbb{R}$ is dense in $\mathbb{R}$ if and only if $\overline{X} = \mathbb{R}$.

Example 4.16. The closed interval $[a, b]$ is a closed set. Indeed, for any sequence $x_n \in [a, b]$, we must have $a \leq \lim x_n \leq b$, hence $\overline{[a, b]} = [a, b]$. Similarly, $\overline{(a, b)} = [a, b]$, since in this case the endpoints aren't in (a, b); but still, we have $a = \lim(a + \frac{1}{n})$ and $b = \lim(b - \frac{1}{n})$.

Example 4.17. Using the density of the rationals in $\mathbb{R}$ we have $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{R} - \mathbb{Q}} = \mathbb{R}$.

Theorem 4.18. *A set $X \subseteq \mathbb{R}$ is closed if and only if X^c is open.*

Proof. X is closed if and only if X^c doesn't contain any adherent points, which is the case if and only if $\forall x \in X^c, \exists \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq X^c$, that is to say, X^c is open. □

Corollary 4.19. $\mathbb{R}$ *itself and* $\emptyset$ *are closed sets.*

Corollary 4.20. *If A and B are closed sets then $A \cup B$ is closed.*

Proof. Notice that $(A \cup B)^c = A^c \cap B^c$ is open. □

Corollary 4.21. *Let $\{A_j\}_{j \in L}$ be a family of closed sets. Then $\bigcap_{j \in L} A_j$ is closed.*

Example 4.22. Arbitrary union of closed sets need not to be closed. For example, for each $x \in (0, 1)$, the set $\{x\}$ is closed since it's finite, but $\bigcup_{x \in (0,1)} \{x\} = (0, 1)$ is open.

Theorem 4.23. *Let $X \subseteq \mathbb{R}$ be an arbitrary set. Then $\overline{X}$ is closed. (i.e. $\overline{\overline{X}} = \overline{X}$)*

Proof. Take $x \in \overline{X}^c$, then we can find an open interval $I \ni x$ such that $I \cap \overline{X} = \emptyset$, hence x in an interior point of $\overline{X}^c$. □

Example 4.24. $\mathbb{R}$ itself is closed, and so is $\emptyset$. Every finite set $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}$ is closed, since its complement is open. Similarly, $\mathbb{Z}$ is closed.

Example 4.25. The sets $\mathbb{Q}$, $\mathbb{R} - \mathbb{Q}$, $(a, b]$, $[a, b)$ are not open nor closed.

Theorem 4.26. *Every set $X \subseteq \mathbb{R}$ has a countable dense subset D, i.e. $\overline{D} = X$.*

Proof. Notice that, if we fix $n \in \mathbb{N}$, we can write $\mathbb{R} = \bigcup_{p \in \mathbb{Z}} \left[\frac{p}{n}, \frac{p+1}{n}\right)$. For each $n \in \mathbb{N}$ and $p \in \mathbb{Z}$ if $X \cap \left[\frac{p}{n}, \frac{p+1}{n}\right) \neq \emptyset$, choose a number $x_{np} \in X \cap \left[\frac{p}{n}, \frac{p+1}{n}\right)$, and let D be the set of all such x_{np}. By construction, D is countable. We claim $\overline{D} = X$. Indeed, let I be an open interval of length $\epsilon > 0$ containing a point $x \in X$. For n sufficiently large such that $\frac{1}{n} < \epsilon$, we can find a $p \in \mathbb{Z}$ such that $\left[\frac{p}{n}, \frac{p+1}{n}\right) \subseteq I$, and hence $x_{np} \in I$. □

A point $a \in \mathbb{R}$ is an *accumulation point* of the set $X \subseteq \mathbb{R}$ if $a = \lim x_n$, for $x_n \in X$ and x_n is sequence with pairwise disjoint elements. Alternatively, every open interval containing a contains points of X other than a itself.

The set of all accumulation points of X is called the *derived set* of X, denoted by X'.

We easily see that if $X' \neq \emptyset$ then X is infinite.

Example 4.27. Let $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Then $X' = \{0\}$.

Example 4.28. $(a, b)' = [a, b]$. Also, $\mathbb{Q}' = (\mathbb{R} - \mathbb{Q})' = \mathbb{R}' = \mathbb{R}$, whereas $\mathbb{Z}' = \emptyset$.

Given a point $a \in \mathbb{R}$ and a set $X \subseteq \mathbb{R}$. We say a is an *isolated point* of X if a is not an accumulation point. In other words, a is isolated if we can find an open interval $I \ni a$ such that $I \cap X = \{a\}$.

Example 4.29. Every natural number $n \in \mathbb{N}$ is isolated. More generally, every $n \in \mathbb{Z}$ is isolated.

Theorem 4.30. *For every $X \subseteq \mathbb{R}$, we have*

$$\overline{X} = X \cup X'.$$

Proof. Since $X \subseteq \overline{X}$ and $X' \subseteq \overline{X}$, we have $X \cup X' \subseteq \overline{X}$. Conversely, let $a \in \overline{X}$. Then every open interval I containing a also contains points of X, either a itself or a point different from a, hence $a \in X \cup X'$. □

Corollary 4.31. *A set X is closed if and only if $X' \subseteq X$.*

Corollary 4.32. *If all the points of X are isolated then X is countable.*

Proof. Let D be a countable dense subset of X, i.e. $\overline{D} = X$, and $x \in X$. By definition, any interval containing x contains points of D, since x is isolated, that can only happen if $x \in D$. Hence $X = D$. □

We need the following lemma to prove the next theorem.

Lemma 4.33. *Let $X \subseteq \mathbb{R}$ be a closed nonempty set with no isolated points. Then $\forall x \in \mathbb{R}$, $\exists I_x \subseteq X$, a closed bounded nonempty subset with no isolated points, such that $x \notin I_x$.*

Proof. Since X is infinite, we can find a point $y \in X$, with $y \neq x$. Take a interval $(a, b) \subseteq \mathbb{R}$ such that $x \notin [a, b]$ and $y \in (a, b)$. Set $A = (a, b) \cap X$, then $A \subseteq X$ is bounded and nonempty. The set $I_x = \overline{A}$ satisfies the desired properties. □

Theorem 4.34. *Let $X \subseteq \mathbb{R}$ be a nonempty closed set such that $X' = X$ (X has no isolated points). Then X is uncountable.*

Proof. The proof is based on lemma 4.33 applied inductively in the following way: Let $\{x_1, x_2, \ldots\}$ be any countable subset of X. We use the lemma to find $I_1 \subseteq X$ such that $x_1 \notin I_1$, and proceed inductively by finding $I_n \subseteq I_{n-1}$ such that $x_n \notin I_n$. Choose $y_n \in I_n$ for each n. Then the sequence y_n is bounded, by Bolzano-Weierstrass theorem, it has a converging subsequence, say $y_{n_k} \to y$. For n sufficiently large we have $y \in I_n$, hence $y \in I_n$ for every $n \in \mathbb{N}$, since the I_n are nested, and moreover $y \neq x_n$ by construction. We conclude that it's impossible for X to be $\{x_1, x_2, \ldots\}$, a countable set. □

Corollary 4.35. *(The contrapositive version) If X is a closed countable nonempty set then X has an isolated point.*

4.3 The Cantor set

The Cantor set is a bounded set $K \subseteq [0, 1]$ defined in the following way: Start with the interval $[0, 1]$ and remove the middle third open interval $(\frac{1}{3}, \frac{2}{3})$. We are left with $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Proceed inductively, removing the middle third of each interval obtained in the previous interation, what is left is the Cantor set K.

For example, the numbers $\frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \ldots$ which are endpoints of removed intervals in each iteration are elements of the Cantor set K. So K has a countable subset. Interesting enough, those are not the only points of K, as a matter of fact most points of K are not endpoints of removed intervals, and it turns out the K is actually uncountable as we shall see.

Since in each iteration we remove a finite amount of intervals, the number of intervals removed is countable. If we denote each open interval removed by I_j, then

$$K = [0,1] - \bigcup_{j=1}^{\infty} I_j = [0,1] \cap \left(\mathbb{R} - \bigcup_{j=1}^{\infty} I_j \right).$$

Since K is the union of two closed sets, it is closed.

Lemma 4.36. *K doesn't have interior points, i.e. int$(K) = \emptyset$.*

Proof. K doesn't have any open intervals, because after each interaction the remaining intervals shrink, so it's impossible to exists an interval $I \subseteq K$ of length l, for any $l \in \mathbb{R}$. Hence, K doesn't have interior points. □

Lemma 4.37. *Let R be the set of endpoints of removed intervals in each iteration. Then R is dense in K, i.e. $\overline{R} = K$.*

Proof. We have to show that given any $x \in K$, for every $\epsilon > 0$, we must have $(x-\epsilon, x+\epsilon) \cap R \neq \emptyset$. If $\epsilon > \frac{1}{2}$, the result is immediate, so let's assume $\epsilon \leq \frac{1}{2}$. At least one of intervals, $(x-\epsilon, x]$ or $[x, x+\epsilon)$, is entirely contained in $[0,1]$, say $(x-\epsilon, x]$. After the n-th iteration, only intervals of length $\frac{1}{3^n}$ are left, hence when $\frac{1}{3^n} < \epsilon$, part of $(x-\epsilon, x]$ will be removed (or was removed already previously), and it can't be the whole $(x-\epsilon, x]$ because $x \in K$. Hence, the endpoint of the removed interval is the point of R we are looking for. □

Corollary 4.38. *K is uncountable.*

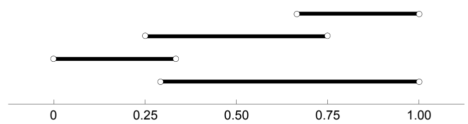
Proof. It follows directly from lemma 4.37 and theorem 4.34. □

4.4 Compact Sets

A open *cover* of a set $X \subseteq \mathbb{R}$ is a collection $C = \{U_j\}_{j \in L}$ (not necessarily countable) of open sets $U_j \subseteq \mathbb{R}$, such that $X \subseteq \bigcup_{j \in L} U_j$. A *subcover* C' of C is a collection formed by sub-indexes $L' \subseteq L$, that is, $C' = \{U_j\}_{j \in L'}$, such that $X \subseteq \bigcup_{j \in L'} U_j$.

A set $X \subseteq \mathbb{R}$ is called **compact**, if every open cover has a finite subcover, that is to say, we can take L' a finite set in the definition above.

Example 4.39. Let $X = (\frac{7}{24}, 1)$. The sets $U_1 = (0, \frac{1}{3}), U_2 = (\frac{1}{4}, \frac{3}{4}), U_3 = (\frac{2}{3}, 1)$ form a (finite) open cover of X, since $X \subseteq U_1 \cup U_2 \cup U_3$. Also, $U_2 = (\frac{1}{4}, \frac{3}{4})$ and $U_3 = (\frac{2}{3}, 1)$ form a subcover, since it is still true that $X \subseteq U_2 \cup U_3$



Example 4.40. Consider the set $X = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, which has all of its points isolated, so it's possible to find an open interval I_n around each point $\frac{1}{n} \in X$, such that $I_n \cap \{\frac{1}{n}\} = \{\frac{1}{n}\}$. Therefore, $C = \{I_n\}_{n \in \mathbb{N}}$ forms an open cover of X, and moreover, C doesn't have any open subcover, since if we remove at least one I_n of C, it ceases to be a cover in the first place.

Theorem 4.41. *(Borel-Lebesgue Theorem – simple version) Any closed interval $[a, b] \subseteq \mathbb{R}$ is compact.*

Proof. We need to prove that any open cover $C = \{I_j\}_{j \in L}$ of $[a, b]$ has a finite subcover. We may assume that I_j are open intervals, since each I_j is open, so it has to contain an interval around each point.

Let X be the set of all points $x \in [a, b]$ such that $[a, x]$ can be cover be finitely many I_j. Notice that $X \neq \emptyset$, since $a \in X$. Set $c = \sup X$, we claim $c = b$. First, we prove $c \in X$. Indeed, $c \leq b$, so we can find $I_{j_0} = (a_0, b_0)$ covering c. Since $c > a_0$, we can find $a_0 < x \leq c$ such that $[a, x] \subseteq I_1 \cup \ldots \cup I_n$, but then $[a, c] \subseteq I_1 \cup \ldots \cup I_n \cup I_{j_0}$, hence $c \in X$. If $c < b$, then we can find $c' \in I_{j_0}$ such that $c < c' < b$. But then $[a, c']$ would still be covered by $I_1 \cup \ldots \cup I_n \cup I_{j_0}$, and c isn't an upper bound, a contradiction. □

Corollary 4.42. *(Borel-Lebesgue Theorem – classical version) Any bounded and closed set $X \subseteq \mathbb{R}$ is compact.*

Proof. Since X is closed, its complement $X^c = \mathbb{R} - X$ is open. Moreover, we can find $[a, b] \supseteq X$, because X is also bounded. Let $C = \{I_j\}_{j \in L}$ be a open cover of X, then $C \cup X^c$ is an open cover of $[a, b]$, by the theorem above we can extract $I_{j_1} \cup \ldots \cup I_{j_n} \cup X^c$, a finite subcover of $[a, b]$. Thus $I_{j_1} \cup \ldots \cup I_{j_n}$ is a finite subcover of X. □

Example 4.43. The real line $\mathbb{R}$ is not compact. Indeed, consider the cover $\mathbb{R} = \bigcup_{n=1}^{\infty} (-n, n)$. Any finite subcover would be equal to the largest interval since they are nested, and hence can't cover the whole line. Similarly, $(0, 1]$ is not compact either, if we consider the nested cover $\bigcup_{n=1}^{\infty} (\frac{1}{n}, 2)$, we can argue like before.

Theorem 4.44. *(Heine–Borel theorem) Let $K \subseteq \mathbb{R}$. The following are equivalent:*

1. *K is closed and bounded;*
2. *K is compact;*
3. *Every infinite subset of K has an accumulation point in K;*
4. *(Sequential compactness) Every sequence $x_n \in K$ has a convergent subsequence with limit in K.*

Proof. We already know that $1 \Rightarrow 2$. We first prove $2 \Rightarrow 3$. It's easy to show the contrapositive of 3, namely, if $X \subseteq K$ doesn't have accumulation points in K then X is finite. Indeed, we can find for each $x \in K$ an interval I_x such that $I_x \cap X = \emptyset$ if $x \notin X$, and $I_x \cap X = \{x\}$ if $x \in X$. Then $\bigcup I_x$ is a cover of K, by compactness, we extract a finite subcover, say $I_{x_1} \cup \ldots I_{x_n}$, but this would force $X = \{x_1, \ldots, x_n\}$,i.e. X is finite.

We now show $3 \Rightarrow 4$. Consider the set $X = \{x_1, x_2, \ldots\}$ formed by elements of the sequence $x_n \in K$. If X is finite then at least one member of the sequence repeat itself infinitely many times, hence forms a constant (convergent) subsequence. Otherwise, by hypothesis we have some $a \in X'$ that is also in K. Equivalently, every neighborhood of $a \in K$ contains point of the sequence x_n, hence a subsequence of x_n converges to a.

Finally, we show $4 \Rightarrow 1$. The proof is by contradiction, namely, suppose K is not bounded or not closed. If K is not closed, at least one sequence x_n converges to a point outside K, so any subsequence of this sequence would also converge to point not in K, a contradiction. If K is not bounded we can easily construct an unbounded sequence, say K is unbounded from above, then construct a sequence satisfying $x_n + 1 < x_{n+1}$, and any subsequence would also be increasing and unbounded, hence can't converge. □

Corollary 4.45. *(Bolzano-Weierstrass alternative version) Every infinite bounded set $X \subseteq \mathbb{R}$ has an accumulation point.*

Proof. Apply theorem 4.44 to $\overline{X}$. □

Corollary 4.46. *Let $K_1 \supseteq K_2 \supseteq \ldots$ be a nested sequence of nonempty compact sets. Then $\bigcap_{j=1}^{\infty} K_j$ is compact and nonempty.*

Example 4.47. The Cantor set K is compact since it's closed and bounded. Every finite set is compact. $\mathbb{Z}$ is not compact because it's unbounded, nor is $\mathbb{R}$ itself. $\mathbb{Q} \cap [0, 1]$ is bounded but it's not compact because it's not closed.

Exercises

1. Show the following: *A set $X \subseteq \mathbb{R}$ is open if and only if for every sequence x_n converging to $a \in A$, $x_n \in A$ for n sufficiently large.*
2. Let $X \subseteq \mathbb{R}$ be open. Show that if $a \in \mathbb{R}$, then $a + X$ is also open, where $a + X = \{a + x; x \in X\}$.
3. Show that $\text{int}(X \cap Y) = \text{int}(X) \cap \text{int}(Y)$, but in general $\text{int}(X \cup Y) \neq \text{int}(X) \cup \text{int}(Y)$. Given an example which illustrates the latter fact.
4. Let A be open and $a \in A$. Show that $A - \{a\}$ is open as well.
5. Show that every collection of nonempty open sets, pairwise disjoints, is countable.
6. Show that the set of accumulation points of a sequence is closed.
7. Let C be closed and $X \subseteq C$. Show that if C is closed then $\overline{X} \subseteq C$.
8. If $\lim x_n = a$ and $X = \{x_1, x_2, \ldots\}$, show that $\overline{X} = X \cup \{a\}$.
9. Let I be a closed interval and suppose $I = A \cup B$, where A, B are closed and disjoints. Show that either $A = I$ or $B = I$.
10. Show that $\frac{1}{4}$ is an element of the Cantor set K. *[Hint: Convince yourself that $\frac{1}{4}$ is an accumulation point]*
11. Let $X \subseteq \mathbb{R}$ be countable. Construct a sequence whose accumulation points is the set $\overline{X}$. Use this to show that every closed set is the set of all accumulation points of a sequence. *[Hint: Write $\mathbb{N}$ as a countable union of infinite disjoints subsets.]*
12. Let K denote the Cantor set. Show that $[0, 1] = \{|x - y|; x, y \in K\}$. *[Hint: Use the fact that proper fractions whose denominator are power of 3 are dense in* $[0, 1]$.*]*
13. Given any $\alpha > 0$. Show that we can find elements $x_1, x_2, \ldots, x_n$ of the Cantor set such that $\alpha = x_1 + x_2 + \ldots + x_n$. *[Hint: Use exercise 12.]*
14. Show that $\overline{X \cup Y} = \overline{X} \cup \overline{Y}$, but in general $\overline{X \cap Y} \neq \overline{X} \cap \overline{Y}$. Given an example which illustrates the latter fact.
15. Give an example of nested sequence $F_1 \supset F_2 \supset \ldots$ of closed nonempty sets such that $\bigcap_j F_j = \emptyset$.
16. Show that a set X is dense in $\mathbb{R}$ if and only if X^c has empty interior.
17. Give an example of open set A such that $\mathbb{Q} \subseteq A$ and $\mathbb{R} - A$ is uncountable.
18. Given an example of an uncountable closed set containing only transcendental numbers. *[Hint: Use exercise 17.]*
19. Given a nonempty set $X \subseteq \mathbb{R}$ and point $a \in \mathbb{R}$, we define *the distance* of a to X as the number $d(a, X) = \inf\{|x - a|; x \in X\}$. Show that

    1. $d(a, X) = 0 \iff a \in \overline{X}$
    2. If X is closed then we can find $b \in X$ such that $d(a, X) = |a - b|$

20. Show that if X is bounded from above then $\overline{X}$ is as well. Moreover, show that $\sup X = \sup \overline{X}$. Prove the equivalent result for $\inf \overline{X}$.
21. Show that if X is bounded then $\sup X$ and $\inf X$ are adherent points.
22. Show that for every $X \subseteq \mathbb{R}$, the derived set X' is closed.

23. Show that a is an accumulation point of X if and only if it is an accumulation point of $\overline{X}$.
24. Show that $(X \cup Y)' = X' \cup Y'$.
25. Let $X \subseteq \mathbb{R}$ be an open set. Show that every point of X is an accumulation point of X.
26. Let $X \subseteq \mathbb{R}$ be a closed set and $a \in X$. Show that a is an isolated point if and only if $X - \{a\}$ is closed.
27. Explain the meaning of the following sentences. You can't use the words in *italic* in your explanation.

    a. $a \in X$ is *not* an *interior* point of X;
    b. $a \in \mathbb{R}$ is *not* an *adherent* point of X;
    c. $X \subseteq \mathbb{R}$ is *not* an *open* set;
    d. $X \subseteq \mathbb{R}$ is *not* a *closed* set;
    e. $a \in \mathbb{R}$ is *not* an *accumulation* point of X;
    f. $X' = \emptyset$;
    g. $X \subseteq Y$ but X is *not dense* in Y;
    h. $int(\overline{X}) = \emptyset$;
    i. $X \cap X' = \emptyset$;
    j. $X \subseteq \mathbb{R}$ is *not* a *compact* set;

28. (Lindelof Theorem) Let $X \subseteq \mathbb{R}$. Any open cover of X has a countable subcover.
29. Let $X \subseteq \mathbb{R}$ be an infinite closed countable set. Show that X has infinitely many isolated points.
30. Show that every real number is the limit of a sequence of pairwise disjoint transcendental numbers.
31. Show that if X is uncountable then $X \cap X' \neq \emptyset$.
32. Obtain an open cover of $\mathbb{Q}$ that doesn't have a finite subcover. Do the same for $[0, +\infty)$.
33. Show that the following are equivalent:

    a. X is bounded;
    b. Every infinite subset of X has an accumulation point (which could be outside X);
    c. Every sequence $x_n \in X$ has a convergent subsequence.

34. (Baire Category Theorem) If $X_1, X_2, X_3, \ldots$ are closed sets with empty interior, then their union $\bigcup_{j=1}^{\infty} X_j$ has empty interior. *[Hint: Use the idea of the proof of theorem 96 from the notes]*
35. Show that $\mathbb{Q}$ is not the intersection of a countable collection of open sets.
36. Let $X \subseteq \mathbb{R}$. Show that if X is uncountable then X' is also.
37. Show that for any $X \subseteq \mathbb{R}$, the set $\overline{X} - X'$ is countable.
38. A point $a \in \mathbb{R}$ is called *condensation point* of X, when every open interval containing a, contains uncountable points of X. Let X_c denotes the set of

all condensation points. Show that X_c is a *perfect set*, i.e. closed with no isolated points, and that $X - X_c$ is countable.

39. (Bendixson theorem) Every closed set $X \subseteq \mathbb{R}$ can be written as a union of a perfect set and a countable set. *[Hint: Use the exercise 38.]*

Chapter 5
Limits

5.1 The limit of a function

Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ be a function of a real variable, and $a \in X'$. We say the number $L \in \mathbb{R}$ is the limit of $f(x)$ as x approaches a, denoted by

$$\lim_{x \to a} f(x) = L,$$

if given $\epsilon > 0$, we can find $\delta > 0$, such that for every $x \in X$:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

In other words, $f(x)$ can be made arbritarily close to L by choosing $x \neq a$ in a sufficiently small neighborhood $(a - \delta, a + \delta)$ of a.

Notice that $a \in X'$ is an accumulation point, so the definition makes sense even if $a \notin X$. In fact, most interesting cases are when $a \notin X$. If a is not an accumulation point, i.e. an isolated point, then the same definition would imply that every number $L \in \mathbb{R}$ is a limit! Hence, the definition only makes sense if $a \in X'$.

Theorem 5.1. *(Uniqueness of limits) Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'$. If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} f(x) = M$, then $L = M$.*

Proof. Given any $\epsilon > 0$, we can find δ, γ such that

$$|x - a| < \delta \Rightarrow |f(x) - L| < \frac{\epsilon}{2}, \text{ and } |x - a| < \gamma \Rightarrow |f(x) - M| < \frac{\epsilon}{2}$$

Let $\alpha = \min\{\delta, \gamma\}$ then

$$|x - a| < \alpha \Rightarrow |L - M| \leq |L - f(x)| + |f(x) - M| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This is only possible if $L - M = 0 \Rightarrow L = M$. ◻

Theorem 5.2. *(Restriction of limits) Let* $Y \subseteq X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$, $a \in X' \cap Y'$. *Consider the restriction* $g : Y \to \mathbb{R}$ *given by* $g(x) = f(x)$ *(Also written as* $f_{|Y}(x)$*). If* $\lim_{x\to a} f(x) = L$ *then* $\lim_{x\to a} g(x) = L$.

Proof. Self-evident. □

Theorem 5.3. *(Local boundedness) If* $\lim_{x\to a} f(x) = L$, *then* $\exists M > 0, \delta > 0$ *such that* $0 < |x - a| < \delta \Rightarrow |f(x)| < M$.

Proof. Take $\epsilon = 1$ in the definition. Then we can find $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < 1 \Rightarrow |f(x)| < |L| + 1 =: M$. □

Theorem 5.4. *(Squeeze-theorem) Let* $X \subseteq \mathbb{R}$, $f, g, h : X \to \mathbb{R}$ *and* $a \in X'$. *If for every* $x \neq a$*:*

$$f(x) \leq g(x) \leq h(x),$$

then

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \Rightarrow \lim_{x\to a} g(x) = L$$

Proof. We can find $\delta, g > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon \Rightarrow L - \epsilon < f(x)$, and $0 < |x - a| < \gamma \Rightarrow |h(x) - L| < \epsilon \Rightarrow h(x) < L + \epsilon$.

Hence, if we set $\alpha = \min\{\delta, \gamma\}$ then $0 < |x - a| < \alpha \Rightarrow L - \epsilon < f(x) \leq g(x) \leq h(x) < L + \epsilon \Rightarrow |g(x) - a| < \epsilon$. □

Theorem 5.5. *(Monotonicity preservation) Let* $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ *and* $a \in X'$. *If* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ *and* $L < M$ *then there exists* $\delta > 0$, *such that* $0 < |x - a| < \delta \Rightarrow f(x) < g(x)$.

Proof. Set $\epsilon := \frac{M-L}{2}$. There exists $\delta > 0$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$ and $|g(x) - M| < \epsilon$. It follows that, $f(x) < \epsilon + L < g(x)$. □

Corollary 5.6. *If* $\lim_{x\to a} f(x) > 0$, *then there exists* $\delta > 0$ *such that* $0 < |x - a| < \delta \Rightarrow f(x) > 0$.

Corollary 5.7. *If* $f(x) \leq g(x)$ *for every* x, *then* $\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$.

Theorem 5.8. *(Equivalent definition of limit) Let* $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ *and* $a \in X'$. *Then* $\lim_{x\to a} f(x) = L$ *if and only if for every sequence* $x_n \in X - \{a\}$, *with* $x_n \to a$, *we have* $\lim_{x\to a} f(x_n) = L$.

Proof. Suppose $\lim_{x\to a} f(x) = L$ and $x_n \to a$. Given $\epsilon > 0$, there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$ and $n > n_0 \Rightarrow 0 < |x_n - a| < \delta$. Therefore, $n > n_0 \Rightarrow |f(x_n) - L| < \epsilon$.

Conversely, suppose $f(x_n) \to L$ for every $x_n \to a$ but $\lim_{x\to a} f(x) \neq L$. There exists $\epsilon > 0$, such that we can find a sequence $x_n \in X - \{a\}$ satisfying $0 < |x_n - a| < \frac{1}{n} \Rightarrow |f(x_n) - L| \geq \epsilon$, but then this sequence converges to a, yet it's not true that $f(x_n) \to L$, a contradiction. □

Corollary 5.9. *(Properties of limits) Let $X \subseteq \mathbb{R}$, $f, g : X \to \mathbb{R}$ and $a \in X'$.*

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a} [f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
3. *Suppose* $\lim_{x\to a} g(x) \neq 0$ *then* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$
4. *Suppose* $\lim_{x\to a} f(x) = 0$ *and* $|g(x)| \leq M$ *then* $\lim_{x\to a} [f(x) \cdot g(x)] = 0$.

Proof. We proved the equivalent result for sequences, the result then follows by theorem 5.8. □

Example 5.10. It follows from the definition of limit that $\lim_{x\to a} x = a$. Similarly, using the properties of limits (Corollary 5.9), we obtain $\lim_{x\to a} x^2 = a^2$. Proceeding by induction, we conclude that $\lim_{x\to a} x^n = a^n$, and hence for every polynomial $p(x) \in \mathbb{R}[x]$, $\lim_{x\to a} p(x) = p(a)$. Similarly, for any rational function $r(x) = \frac{p(x)}{q(x)}$, if $q(a) \neq 0$ then $\lim_{x\to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}$.

Example 5.11. Consider the function:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then for any $a \in \mathbb{R}$, the limit $\lim_{x\to a} f(x)$ doesn't exist. Indeed, given any real number a we can construct two sequences $x_n \in \mathbb{Q}$ and $y_n \in \mathbb{R} - \mathbb{Q}$, with $x_n \to a$ and $y_n \to a$. Therefore, $f(x_n) \to 1$ but $f(y_n) \to 0$, so $\lim_{x\to a} f(x)$ doesn't exist.

Example 5.12. Consider the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ given by $f(x) = \sin(\frac{1}{x})$. We claim $\lim_{x\to 0} f(x)$ doesn't exist. It's enough to find two sequences $x_n \to 0$ and $y_n \to 0$ such that $f(x_n)$ and $f(y_n)$ converge to different limits. Take $x_n = \frac{1}{n\pi}$ and $y_n = (\frac{\pi}{2} + 2n\pi)^{-1}$, then $f(x_n) \to 0$ but $f(y_n) \to 1$.

5.2 One sided and infinite limits

Let $X \subseteq \mathbb{R}$ and $a \in \mathbb{R}$. We say a is *accumulation point to the right* (or one-sided right accumulation point) if for every $\epsilon > 0$, $(a, a+\epsilon) \cap X \neq \emptyset$. Similarly, a is *accumulation point to the left* if for every $\epsilon > 0$, $(a-\epsilon, a) \cap X \neq \emptyset$.

We denote $X'_+ (X'_-)$, the set of all accumulation points to the right (left) of X. The definition of limit can be extended in this scenario as well. For example, let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'_+$, then we write

$$\lim_{x \to a^+} f(x) = L$$

If $\forall \epsilon > 0, \exists \delta > 0,\ 0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$. We define $\lim_{x \to a^-} f(x) = L$ analogously.

Theorem 5.13. *Let $X \subseteq \mathbb{R}$, $f : X \longrightarrow \mathbb{R}$ and $a \in X'$. Then $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$.*

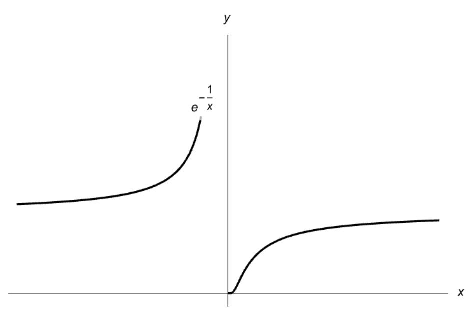
Proof. The conditional implication is trivial, we prove the converse. Suppose $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$. Then we can find $\delta, \gamma > 0$ such that given $\epsilon > 0$, $0 < x - a < \delta \Rightarrow |f(x) - L| < \epsilon$ and $0 < a - x < \gamma \Rightarrow |f(x) - L| < \epsilon$. If we set $\alpha = \min\{\delta, \gamma\}$, then $0 < |x - a| < \alpha \Rightarrow |f(x) - L| < \epsilon$. □

Example 5.14. Consider the function sign : $\mathbb{R} - \{0\} \longrightarrow \mathbb{R}$ given by

$$\text{sign}(x) = \frac{x}{|x|}.$$

Then $\lim_{x \to 0^-} \text{sign}(x) = -1$ but $\lim_{x \to 0^+} \text{sign}(x) = 1$, so $\lim_{x \to 0} \text{sign}(x)$ doesn't exist.

Example 5.15. Consider the function $f(x) : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = e^{-\frac{1}{x}}$.



Then $\lim_{x \to 0^+} f(x) = 0$ but $\lim_{x \to 0^-} f(x)$ doesn't exist.

Recall that a function is *increasing* if $x < y \Rightarrow f(x) < f(y)$, *nondecreasing* if $x \leq y \Rightarrow f(x) \leq f(y)$. We define *decreasing, nonincreasing* in a similar way. Finally we say a function is *monotone* if satisfies any of the above conditions.

Theorem 5.16. *Let $X \subseteq \mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ a bounded monotone function. Given $a \in X'_+, b \in X'_-$, the one sided limits $\lim_{x \to a^+} f(x)$ and $\lim_{x \to b^-} f(x)$ exist.*

Proof. Without loss of generality, suppose $f(x)$ increasing. We prove $\lim_{x \to a^+} f(x)$ exist, the other limit is analogous. Set $L := \inf\{f(x); x > a\}$. We claim $\lim_{x \to a^+} f(x) = L$. Indeed, given $\epsilon > 0$ the number $\epsilon + L$ is not a lower bound, hence we can find $\delta > 0$ such that $L \leq f(a + \delta) < L + \epsilon$. Since $f(x)$ is increasing, it follows that $a < x < a + \delta \Rightarrow L \leq f(x) < L + \epsilon$, as required. □

Let $X \subseteq \mathbb{R}$ be a set unbounded from above. Given $f : X \to \mathbb{R}$ we write

$$\lim_{x\to+\infty} f(x) = L,$$

if there is a number $L \in \mathbb{R}$ such that

$$\forall \epsilon > 0, \exists M > 0,\ M < x \Rightarrow |f(x) - L| < \epsilon.$$

The limit $\lim\limits_{x\to-\infty} f(x)$ is defined analogously. Notice that both infinite limits are, in a way, one sided limits. In particular, the limit of a sequence x_n is an infinite limit when we consider the sequence as a function $x : \mathbb{N} \to \mathbb{R}$, i.e. $\lim x_n = \lim\limits_{n\to+\infty} x(n)$.

Example 5.17. We have $\lim\limits_{x\to-\infty} \frac{1}{n} = \lim\limits_{x\to+\infty} \frac{1}{n} = 0$. Also, $\lim\limits_{x\to-\infty} e^x = 0$ but $\lim\limits_{x\to+\infty} e^x$ doesn't exist.

Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'$. We write

$$\lim_{x\to a} f(x) = +\infty,$$

if $\forall M > 0, \exists \delta > 0,\ 0 < |x - a| < \epsilon \Rightarrow f(x) > M$.

The definition of $\lim\limits_{x\to a} f(x) = -\infty$, $\lim\limits_{x\to\pm\infty} f(x) = \pm\infty$, and $\lim\limits_{x\to a^\pm} f(x) = \pm\infty$ can be given *mutatis mutandis*.

Example 5.18. With the definitions above we have, for example, $\lim\limits_{x\to+\infty} e^x = +\infty$, $\lim\limits_{x\to-\infty} x^2 = +\infty$, $\lim\limits_{x\to 2^-} \left(\frac{1}{x-2}\right) = -\infty$, $\lim\limits_{x\to 2^+} \left(\frac{1}{x-2}\right) = +\infty$.

The theorem below can be proven using the same arguments we used to prove their finite counterpart, so the proof will be ommitted.

Theorem 5.19. *(Properties of infinite limits) Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'$.*

- *(Uniqueness) If $\lim\limits_{x\to a} f(x) = +\infty$ then it's impossible to have $\lim\limits_{x\to a} f(x) = L$ for $L \in \mathbb{R}$ or $L = -\infty$.*
- *(Restriction) If $\lim\limits_{x\to a} f(x) = +\infty$, then for every $Y \subseteq X$, if we set $g(x) = f_{|Y}(x)$, we still have $\lim\limits_{x\to a} g(x) = +\infty$.*
- *(Unboundedness) If $\lim\limits_{x\to a} f(x) = +\infty$, then $f(x)$ is not bounded in any neighborhood of $a \in X$.*
- *(Monotonicity) If $f(x) \le h(x)$ and $\lim\limits_{x\to a} f(x) = +\infty$, then $\lim\limits_{x\to a} h(x) = +\infty$.*
- *(Preservation of the sign) If $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} h(x) = +\infty$, then $\exists \delta > 0$ such that $0 < |x - a| < \delta \Rightarrow f(x) < h(x)$.*
- *(Equivalent definition) $\lim\limits_{x\to a} f(x) = +\infty$ if and only if for every sequence $x_n \in X - \{a\}$ with $\lim x_n = a$, we have $\lim\limits_{n\to\infty} f(x_n) = +\infty$.*

Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ and $a \in X'$. We say f is *bounded in a neighborhood* of a, if there is $k, \delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f(x)| \leq k$$

A number $c \in \mathbb{R}$ is an *adherent value* of f at a if there exists a sequence $x_n \in X$ such that $\lim x_n = a$ and $\lim f(x_n) = c$. In particular, if a function has a limit $\lim\limits_{x \to a} f(x) = L$, then L is the only adherent value.

Given $a \in X'$ and $\delta > 0$, we denote by I_δ the $\delta-$neighborhood around a given by $I_\delta = X - \{a\} \cap (a - \delta, a + \delta)$.

Theorem 5.20. *A number $c \in \mathbb{R}$ is an adherent value of f at a if and only if for every $\delta > 0$ we have $c \in \overline{f(I_\delta)}$.*

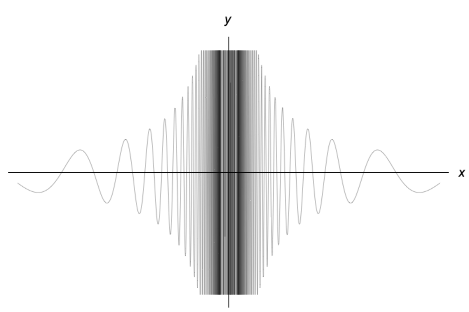
Proof. Suppose $c \in \mathbb{R}$ is an adherent value. Then $a = \lim x_n$ and $c = \lim f(x_n)$. Since $I_\delta \ni a$, $x_n \in I_\delta$ for n sufficiently large, so $f(x_n) \in f(I_\delta)$. Conversely, suppose $c \in \overline{f(I_\delta)}$ for every $\delta > 0$. We can take δ of the form $\delta = \frac{1}{n}$, for $n \in \mathbb{N}$, to obtain a sequence $x_n \in I_{\frac{1}{n}}$, such that $|f(x_n) - c| < \frac{1}{n}$. We conclude that $\lim x_n = a$ and $\lim f(x_n) = c$. □

Let's denote the set of all adherent values at a of a function f by $AV(f, a)$.

Corollary 5.21. $AV(f, a) = \bigcap\limits_{\delta > 0} \overline{f(I_\delta)}$

Corollary 5.22. *$AV(f, a)$ is a closed set. If f is bounded in a neighborhood of a, then $AV(f, a)$ is compact and nonempty.*

Example 5.23. Let $f(x) = \frac{\sin\left(\frac{1}{x}\right)}{x}$, whose graph is shown below.



Every $c \in \mathbb{R}$ is an adherent value of f at 0, that is, $AV(f, 0) = \mathbb{R}$. Indeed, given any $c \in \mathbb{R}$ and an open intervals $(c - \epsilon, c + \epsilon) \ni c$ and $I_\delta := (-\delta, \delta) \ni 0$, we claim $(c - \epsilon, c + \epsilon) \cap f(I_\delta) \neq \emptyset$, or equivalently, $c - \epsilon < \frac{\sin\left(\frac{1}{a}\right)}{a} < c + \epsilon$ for some $a \in (-\delta, \delta)$, which is easily true by the periodicity of $\sin(x)$ and the behavior of $\frac{1}{x}$.

Example 5.24. Let $f(x) = \frac{1}{x}$, then $AV(f, 0) = \emptyset$.

According to corollary 5.22, if f is bounded in a neighborhood of a, the set $AV(f,a) \neq \emptyset$ is compact, hence has a maximum and minimum value.

We call the maximum value of $AV(f,a)$ the *limit superior* of f at a and denote it by

$$\lim_{x\to a} \sup f(x).$$

Similarly, the minimum value of $AV(f,a)$ is called the *limit inferior* of f at a and denote it by

$$\lim_{x\to a} \inf f(x).$$

We use the convention that when f is not bounded around a, we write $\lim_{x\to a} \sup f(x) = +\infty$ and $\lim_{x\to a} \inf f(x) = -\infty$.

Example 5.25. Let $f(x) = \sin\left(\frac{1}{x}\right)$ then $AV(f,0) = [-1,1]$. Indeed, for a fixed $a \in [-1,1]$ consider $x_n = (a + 2\pi n)^{-1}$, then $f(x_n) = a$. Therefore, $\lim_{x\to a} \inf f(x) = -1$ and $\lim_{x\to a} \sup f(x) = 1$.

Theorem 5.26. *Let f be a bounded function in a neighborhood of a. Then given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |x - a| < \delta \Rightarrow \lim_{x\to a} \inf f(x) - \epsilon < f(x) < \lim_{x\to a} \sup f(x) + \epsilon.$$

Corollary 5.27. $\lim_{x\to a} f(x) = L$ *if and only if f has only one adherent value at a, namely L itself.*

5.3 Continuity

Intuitively, a continuous function is a function whose graph has no gaps or holes. More precisely, let $f : X \to \mathbb{R}$ be a real valued function and $a \in X$. We say f is *continuous* at a if

$$\forall \epsilon > 0, \exists \delta > 0;\ |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

If f is continuous for every $a \in X$ we simply say f is continuous.

Notice that if $a \in X$ is an isolated point then any function $f : X \to \mathbb{R}$ is continuous at a. In particular, if $X' = \emptyset$ then any function $f : X \to \mathbb{R}$ is continuous.

Example 5.28. Any function $f : \mathbb{Z} \to \mathbb{R}$ is continuous, since $\mathbb{Z}' = \emptyset$.

Theorem 5.29. *If $a \in X'$, then f is continuous at a if and only if* $\lim_{x\to a} f(x) = f(a)$.

Proof. Self-evident. □

By using the already proven properties of limits we conclude:

Theorem 5.30. *If $f : X \to \mathbb{R}$ is continuous then for any $Y \subseteq X$ the restriction $f_{|Y}$ is also continuous. Conversely, if $Y = I \cap X$ for some open interval I containing a point $a \in X$, then if $f_{|Y}$ is continuous at a, f is also continuous at a.*

In other words, theorem 5.30 says that continuity is a *local property*. More precisely, if f coincides with a continuous function in a neighborhood of $a \in X$, then f itself is continuous at a.

Corollary 5.31. *If f is continuous at $a \in X$, then f is bounded in a neighborhood of a.*

Corollary 5.32. *If f, g are continuous at $a \in X$ and $f(a) < g(a)$, then $f(x) < g(x)$ in a neighborhood of a.*

Corollary 5.33. *If f is continuous at $a \in X$ and $f(a) < k$ $(f(a) > k)$, for some $k \in \mathbb{R}$, then $f(x) < k$ $(f(x) > k)$ in a neighborhood of a.*

Using the alternate definition of limit we can prove:

Theorem 5.34. *f is continuous at $a \in X$ if and only if for every sequence $x_n \to a$, we have $f(x_n) \to f(a)$.*

Theorem 5.35. *f, g are continuous at $a \in X$, them $f + g, f - g$, and $f \cdot g$ are also continuous at a. If $g(a) \neq 0$ then f/g is also continuous at a. Moreover, the composition of continuous function is also continuous.*

Example 5.36. The function $f(x) = x$ is clearly continuous, hence its self-product x^n is also continuous, and so is any polynomial $p(x) = a_n x^n + \ldots + a_1 x + a_0$. A rational function $p(x)/q(x)$ is continuous at points where $q(x) \neq 0$.

Example 5.37. The function $f(x) = |x|$ is continuous on the open interval $(0, +\infty)$ since it is constant there, for the same reason it's also continuous in $(-\infty, 0)$. Finally, it's continuous at 0, since $\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0$. On the other hand, the function defined by $g(x) = \frac{x}{|x|}$, if $x \neq 0$, and $g(0) = 1$, is not continuous at the origin since $\lim_{x \to 0^-} g(x) = -1 \neq \lim_{x \to 0^+} g(x) = 1$.

Theorem 5.38. *Suppose $X \subseteq A \cup B$, where $A, B \subseteq \mathbb{R}$ are closed sets. If the function $f : X \to \mathbb{R}$ satisfies $f_{|X \cap A}$ is continuous and $f_{|X \cap B}$ is continuous, then f itself is continuous.*

Proof. Let $a \in X$ and $\epsilon > 0$ be given. Suppose first $a \in A \cap B$. Then there are $\delta, \gamma > 0$ such that $\forall x \in X \cap A, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ and $\forall x \in X \cap B, |x - a| < \gamma \Rightarrow |f(x) - f(a)| < \epsilon$. Set $\alpha = \min\{\delta, \gamma\}$, then $\forall x \in X, |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \epsilon$, which implies f is continuous at a.

Now suppose $a \in A$ but $a \notin B$. There exists $\delta > 0$, such that $\forall x \in X \cap A, |x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. Since B is closed, $\overline{B} = B$, and we can find $\gamma > 0$ such that $|x - a| < \gamma \Rightarrow x \notin B$. As before, if we set $\alpha = \min\{\delta, \gamma\}$, then $\forall x \in X, |x - a| < \alpha \Rightarrow |f(x) - f(a)| < \epsilon$, as desired. The case $a \notin A$ but $a \in B$ can be proven analogously. □

Corollary 5.39. *Suppose $X = A \cup B$, where $A, B \subseteq \mathbb{R}$ are closed sets. If the restrictions $f_{|A}, f_{|B}$ of a function $f : X \to \mathbb{R}$ are continuous, then f itself is continuous.*

We can generalize the result above if we take the cover $A \cup B$ to be open. In fact, a stronger result is valid. (The proof follows directly from theorem 5.30 and will be omitted.)

Theorem 5.40. *(Sheaf property) Let $X \subseteq \bigcup_{\lambda \in L} A_\lambda$ be an open cover of X. If the restrictions $f_{|X \cap A_\lambda}$ of a function $f : X \to \mathbb{R}$ are continuous, then f itself is continuous*

Corollary 5.41. *Suppose $X = \bigcup_{\lambda \in L} A_\lambda$, where each A_λ is open. If the restrictions $f_{|A_\lambda}$ of a function $f : X \to \mathbb{R}$ are continuous, then f itself is continuous*

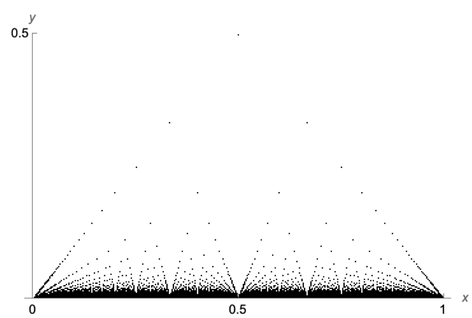
Example 5.42. Consider again $f(x) = \frac{x}{|x|}$ but this time with domain $X = (-\infty, 0) \cup (0, +\infty)$. Then f is continuous by the corollary above.

Let $f : X \to \mathbb{R}$ be a real valued function and $a \in X$. If f is not continuous at a, we say it is *discontinuous* at a.

Example 5.43. (Thomae's function)The function $f : \mathbb{R} \to \mathbb{R}$ given by:

$$f(x) = \begin{cases} \frac{1}{q}, & \text{if } x \in \mathbb{Q} \text{ and } x = \frac{p}{q}, p \in \mathbb{Z}, q \in \mathbb{N}, \gcd(p, q) = 1 \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

The graph of $f(x)$ on the interval $(0, 1)$ is shown below.



Notice that $f(x)$ is periodic, since $f(x + 1) = f(x)$. We claim that f is discontinuous at any $a \in \mathbb{Q}$. Indeed, we can find a sequence, say $x_n = a + \frac{\sqrt{2}}{n}$, of irrational numbers, with $x_n \to a$ but $f(x_n) \to 0$, since $f(a) \neq 0$ in this case, f can't be continuous at a.

Surprisingly enough, f is continuous at every $a \notin \mathbb{Q}$. Equivalently, we must have $\lim_{x \to a} f(x) = 0$. Since f is periodic, it's enough to prove the continuity for $a \in (0, 1) \cap (\mathbb{R} - \mathbb{Q})$.

Suppose $\epsilon > 0$ is given. Using the Archimedean property of $\mathbb{R}$, there is $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. Decompose $(0, 1)$ into k subintervals of length $\frac{1}{k}$, for $k = 1, 2, \ldots, n$. Then 'a' will be in one of these intervals, for each k, say $a \in (\frac{m_k}{k}, \frac{m_k+1}{k})$. Let $\delta_k = \min\left\{|a - \frac{m_k}{k}|, |a - \frac{m_k+1}{k}|\right\}$, the minimum distance between a and the endpoints of $(\frac{m_k}{k}, \frac{m_k+1}{k})$, and define $\delta := \min\limits_{1 \le k \le n} \delta_k$.

Given $x \in (a - \delta, a + \delta)$ if $x \notin \mathbb{Q}$ then $f(x) = 0 < \epsilon$. Otherwise, $x = \frac{p}{q}$ and by minimality of δ, we must have $q > n$, hence $f(x) = \frac{1}{q} < \frac{1}{n} < \epsilon$ and we conclude that $\lim\limits_{x \to a} f(x) = f(a) = 0$.

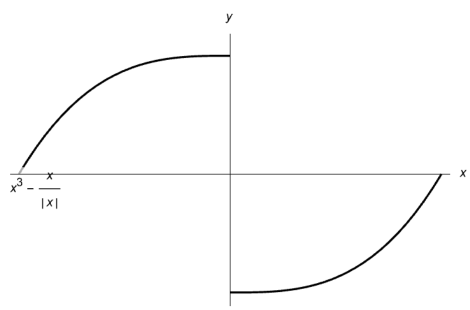
It's impossible to have a function which is discontinuous at every irrational number, see the exercises.

Example 5.44. If $f : \mathbb{R} \to \mathbb{R}$ is given by:

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{if } x \in \mathbb{R} - \mathbb{Q} \end{cases}$$

Then f is discontinuous at every $a \in \mathbb{R}$, since the limit $\lim\limits_{x \to a} f(x)$ doesn't exist.

Example 5.45. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(0) = 1$ and $f(x) = x^3 - \frac{x}{|x|}$ if $x \neq 0$. Then f is discontinuous at 0 only.



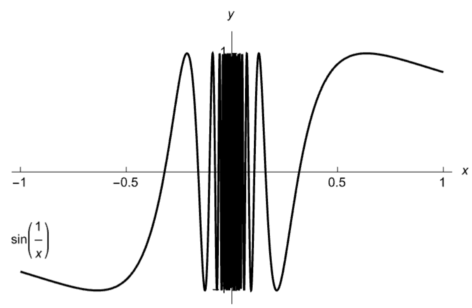
Example 5.46. Let K be the Cantor set. Consider the function $f : [0, 1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \in K \\ 1, & \text{if } x \notin K \end{cases}$$

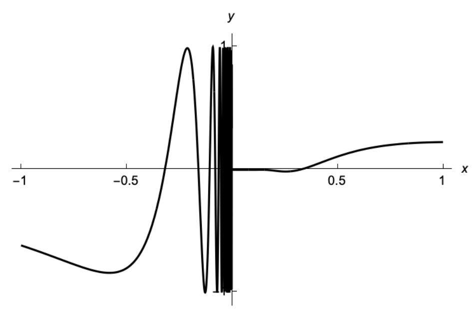
Then f is discontinuous at every point $a \in K$ and continuous at the open set K^c. Indeed, f is constant, hence continuous, at every $a \in K^c$.

Suppose now $a \in K$. Since every point of K is an accumulation point, it's possible to find a sequence $x_n \notin K$ such that $x_n \to a$, hence $f(x_n) \to 1 \neq 0$, so f is discontinuous at a.

Example 5.47. The function $f(0) = a$ and $f(x) = \sin \frac{1}{x}$ if $x \neq 0$ is discontinuous at 0, regardless of $a \in \mathbb{R}$, since $\lim\limits_{x \to 0} f(x)$ doesn't exist.



Example 5.48. The function $f(0) = 0$ and $f(x) = \frac{\sin\frac{1}{x}}{1+e^{\frac{1}{x}}}$ if $x \neq 0$ is discontinuous at 0, since $\lim_{x\to 0^-} f(x)$ doesn't exist. In this case, $\lim_{x\to 0^+} f(x) = 0$ however.



Example 5.49. The function $f(0) = 0$ and $f(x) = \frac{1}{1+e^{\frac{1}{x}}}$ if $x \neq 0$ is discontinuous at 0, since $\lim_{x\to 0^-} f(x) = 1$ but $\lim_{x\to 0^+} f(x) = 0$.

Let $f : X \to \mathbb{R}$, $a \in X$ and suppose f is discontinuous at a. Then we say $a \in X$ is a *jump discontinuity*, if both one sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exists but are different. If at least one of the one sided limits doesn't exist, then we say $a \in X$ is an *essential discontinuity*.

Theorem 5.50. *A monotone function $f : X \to \mathbb{R}$ can't have essential discontinuities.*

Proof. Suppose f nondecreasing and $a \in X$. If $x + \delta \in X$ then f is bounded in $[x, x + \delta] \cap X$. The result then follows from theorem 5.16. □

Theorem 5.51. *Let $f : X \to \mathbb{R}$ be a function having only jump discontinuities. Then the set of discontinuities of f is countable.*

Proof. Define the jump function $j(x) : X \to \mathbb{R}$ of f by:

$$j(a) = \begin{cases} 0, \text{ if } a \text{ is isolated.} \\ |f(a) - \lim\limits_{x\to a^+} f(x)|, \text{ if } a \in X'_+ \text{ only.} \\ |f(a) - \lim\limits_{x\to a^-} f(x)|, \text{ if } a \in X'_- \text{ only.} \\ \max\{|f(a) - \lim\limits_{x\to a^+} f(x)|, |f(a) - \lim\limits_{x\to a^-} f(x)|\}, \text{ if } a \in X'_+ \cap X'_-. \end{cases}$$

Intuitively, $j(x)$ measures the length of the 'jump' of $f(x)$. Consider the set

$$C_n := \{x \in X; j(x) \geq \frac{1}{n}\}.$$

The set of discontinuities of $f(x)$ is the set $\bigcup\limits_{n=1} C_n$, hence if we can prove that each C_n is countable then we're done. We claim that for each $n \in \mathbb{N}$, the set C_n has only isolated points, hence it's countable (see corollary 4.32).

Let $a \in C_n$ and suppose $a \in X'_+$. By using the definition of one sided limit, if we set $L := \lim\limits_{x\to a^+} f(x)$ we can find $\delta > 0$ such that $0 < x-a < \delta \Rightarrow |f(x)-L| < \frac{1}{4n} \Rightarrow L - \frac{1}{4n} < f(x) < L + \frac{1}{4n}$, hence if $x \in (a, a+\delta)$ then $j(x) \leq \frac{1}{2n}$, which is to say $(a, a+\delta) \cap C_n = \emptyset$. If $a \notin X'_+$, we can just choose $\delta > 0$ such that $(a, a+\delta) \cap X = \emptyset$. In any case, we can find $\delta > 0$ such that $(a, a+\delta) \cap C_n = \emptyset$. A similar argument implies we can find $\gamma > 0$ such that $(a-\gamma, a) \cap C_n = \emptyset$. We conclude that $a \in C_n$ is isolated. □

Corollary 5.52. *The set of discontinuities of a monotone function f is countable.*

5.4 Continuous functions defined on intervals

The next result highlights the fact that continuous functions can't have gaps, in other words, if two numbers $a \neq b$ are in the range, then $[a, b]$ is also in the range.

Theorem 5.53. *(Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $d \in \mathbb{R}$ be a number such that $f(a) < d < f(b)$. Then there is $c \in [a, b]$ such that $d = f(c)$.*

Proof. Define $X = \{x \in [a, b]; f(x) < d\}$. This set is nonempty because $f(a) < f(d)$, and due to the continuity of $f(x)$, X doesn't have a maximum element. Set $c = \sup X$, then $c \notin X$. However, since c is an adherent value, there is a sequence $x_n \to c$, which implies $f(c) \leq d$. We conclude that $f(c) = d$. □

Corollary 5.54. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval (not necessarily bounded). If $a, b \in I$ and $f(a) < d < f(b)$, then there exists $c \in I$ such that $f(c) = d$.*

Corollary 5.55. *Let $f : I \to \mathbb{R}$ be a continuous function, where I is an interval. Then $f(I)$ is an interval.*

Proof. If we set $c = \inf f(x)$ and $d = \sup f(x)$ then $f(I)$ is an interval with endpoints c and d *(not necessarily bounded, nor open/closed).* $\square$

Example 5.56. Let $f : I \to \mathbb{R}$ be a continuous function such that $f(I) \subseteq Y$, where Y has empty interior. Then f is constant. Indeed, it follows by 5.55 that $f(I)$ is an interval, so it must be of the form $[c, c]$, otherwise, $f(I)$ would have an interior point. In particular, every continuous function $f : I \to \mathbb{Z}$ is constant.

Example 5.57. Every polynomial $p(x) = a_{2n-1}x^{2n-1} + \ldots + a_0$ of odd degree has at least one real root. Indeed, in this case $p(x)$ is a continuous function defined on the interval $(-\infty, +\infty)$, so its image is an interval. Since $\lim_{x\to\pm\infty} p(x) = \pm\infty$, that interval has to be $(-\infty, +\infty)$, hence $p(x)$ is surjective.

A function $f : X \to Y$ is a *homeomorphism*, if f is a continuous bijection having a continuous inverse f^{-1}.

Theorem 5.58. *Let $f : I \to \mathbb{R}$ be a continuous injective function defined on a interval I. Then f is monotone, and if we set $J = f(I)$, then $f : I \to J$ is a homeomorphism.*

Proof. It's enough to prove the result for $I = [a, b]$. Suppose $f(a) < f(b)$, we claim f is increasing. Suppose not, that is, we can find $c, d \in [a, b]$ such that $c < d$ but $f(c) > f(d)$. Either $f(a) < f(d)$ or $f(a) > f(d)$. If $f(a) < f(d) < f(c)$, by theorem 5.53, we can find $p \in (a, c)$ such that $f(p) = f(d)$, a contradiction by the injectivity of f. For the same reason we can't have $f(d) < f(a) < f(b)$. Hence, f has to be increasing.

Using corollary 5.55, we see that J is an interval, hence $f^{-1} : J \to I$ is an increasing function (since f is) whose image is an interval. Suppose f^{-1} is not continuous at a point $y \in J$, say $M := \lim_{x\to y^+} f^{-1}(x) \neq L := \lim_{x\to y^-} f^{-1}(x)$. Then $f^{-1}(c) \in (L, M)$ and $(L, M) \cap I = \{f^{-1}(c)\}$, which implies I has an isolated point, a contradiction. $\square$

Theorem 5.59. *Let $f : X \to \mathbb{R}$ be a continuous function. If X is compact then $f(X)$ is compact.*

Proof. We claim $f(X)$ is sequentially compact, which is equivalent to compactness by theorem 4.44. Let $y_n = f(x_n)$ be a sequence in $f(X)$, we claim it has a converging subsequence. By the compactness of X, there is a converging subsequence $x_{n_k} \to x \in X$. If we set $y_{n_k} = f(x_{n_k})$, then $y_{n_k} \to f(x)$, since f is continuous. $\square$

Corollary 5.60. *(Weierstrass Extreme Value Theorem) Let $X \subseteq \mathbb{R}$ be compact and $f : X \to \mathbb{R}$ be a continuous function. Then f achieves its maximum and minimum value, that is to say, there are $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.*

Theorem 5.61. *Let $X \subseteq \mathbb{R}$ be compact and $f : X \to \mathbb{R}$ be a continuous injective function. If we set $Y := f(X)$, then $f : X \to Y$ is a homeomorphism.*

Proof. Let $y \in Y$, we claim f^{-1} is continuous at $y = f(x)$. Suppose $y_n = f(x_n)$ is a sequence of points in Y such that $y_n \to y = f(x)$, we claim $x_n \to x$. It's enough to prove that any converging subsequence of x_n converges to x. Let x_{n_k} be a converging subsequence, say $x_{n_k} \to a \in X$. Then $y_{n_k} \to f(a)$, but since y_{n_k} is a subsequence of y_n, it also converges to $f(x)$, by the injectivity of f we deduce that $a = x$. □

We say a function $f : X \to \mathbb{R}$ is uniformly continuous if

$$\forall \epsilon > 0, \exists \delta > 0 : \forall x, y \in X, |x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

It follows that every uniformly continuous function is continuous. The converse is false, as the example below illustrates.

Example 5.62. The function $f(x) = \frac{1}{x}$ is continuous on $(0, +\infty)$ but is not uniformly continuous. Indeed, given $\epsilon, \delta > 0$, take a point $0 < x < \min\{\delta, \frac{1}{3\epsilon}\}$ and $y = x + \frac{\delta}{2}$. Then $|x - y| < \delta$ but

$$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{x + \frac{\delta}{2}}\right| = \left|\frac{\delta}{x(2x + \delta)}\right| > \left|\frac{\delta}{3\delta x}\right| > \epsilon.$$

Example 5.63. Linear functions $f(x) = mx + b$ are continuous. Indeed, given $\epsilon > 0$ just take $\delta = \frac{\epsilon}{|m|}$, so that $|x - y| < \delta \Rightarrow |f(x) - f(y)| = |m(x - y)| \leq |m|\frac{\epsilon}{|m|} = \epsilon$.

Example 5.64. A function $f : X \to \mathbb{R}$ is called *Lipschitz* if there exists a constant $C > 0$ such that $|f(x) - f(y)| \leq C|x - y|$. Any Lipschitz function is obviously uniformly continuous. For example, linear functions $f(x) = mx + b$ are Lipschitz, and if X is bounded, $f(x) = x^n$ is Lipschitz.

Theorem 5.65. *If $f : X \to \mathbb{R}$ is uniformly continuous and x_n is a Cauchy sequence then $f(x_n)$ is also Cauchy.*

Corollary 5.66. *If $f : X \to \mathbb{R}$ is uniformly continuous and $a \in X'$ then $\lim_{x \to a} f(x)$ exists.*

Example 5.67. The functions $f(x) = \sin \frac{1}{x}$ and $g(x) = \frac{1}{x}$ can't be uniformly continuous because the limit when when x approaches 0 doesn't exist.

Theorem 5.68. *Let $X \subseteq \mathbb{R}$ be compact and $f : X \to \mathbb{R}$ continuous then f is uniformly continuous.*

Exercises

1. Consider the following typo in the definition of limit:

$$\forall \epsilon > 0, \exists \delta > 0, x \in X,\ 0 < |x - a| < \epsilon \Rightarrow |f(x) - L| < \delta.$$

Show that f satisfies this condition if and only if it is bounded around each interval centered in $a \in X$. In the affirmative case, L can be any real number.
2. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be given by

$$\frac{1}{1 + e^{\frac{1}{x}}}.$$

Compute $\lim_{x \to 0^-} f(x)$ and $\lim_{x \to 0^+} f(x)$.
3. Let $f(x) = x + 10 \sin x$. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.
4. Let $f : X \to \mathbb{R}$ be a monotone function. Show that the set of points $a \in X'$ such that $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ is countable.
5. Let $a > 1$ and $f : \mathbb{Q} \to \mathbb{R}$ given by $f(\frac{p}{q}) = a^{\frac{p}{q}}$. Show that $\lim_{x \to 0} f(x) = 1$.
6. Let $a > 1$ and $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$. Show that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = 0$
7. Let $p(x) \in \mathbb{R}[x]$ be a polynomial. If the leading coefficient is positive, show that $\lim_{x \to +\infty} p(x) = +\infty$.
8. Find the set of adherent points at 0 of the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be given by $f(x) = \frac{\sin(\frac{1}{x})}{1+e^{\frac{1}{x}}}$
9. If $\lim_{x \to a} f(x) = L$, show that $\lim_{x \to a} |f(x)| = |L|$, and that the set of adherent points at a is $\{L\}, \{-L\}$ or $\{-L, L\}$.
10. Given a nonempty compact set $K \subseteq \mathbb{R}$ and a point $a \in \mathbb{R}$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ whose the set of adherent points at a is K.
11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x, & x \notin \mathbb{Q} \\ 0, & x = 0 \\ q, & x = \frac{p}{q} \text{ and } gcd(p, q) = 1, p > 0 \end{cases}$$

Show that f is unbounded in any non-degenerate interval.
12. Recall that the floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ is given by $\lfloor x \rfloor :=$ largest integer less than or equal to x. Show that if $a, b \in R$ are positive numbers then

$$\lim_{x \to 0^+} \frac{x}{a} \left\lfloor \frac{b}{x} \right\rfloor = \frac{b}{a} \text{ and } \lim_{x \to 0^+} \frac{b}{x} \left\lfloor \frac{x}{a} \right\rfloor = 0$$

13. Let $f, g : X \to \mathbb{R}$ be functions bounded in a neighborhood of $a \in X'$. Show that
$$\lim_{x\to a} \sup(f+g) \leq \lim_{x\to a} \sup f(x) + \lim_{x\to a} \sup g(x),$$
and also that
$$\lim_{x\to a} \sup(-f(x)) = -\lim_{x\to a} \inf f(x)$$
14. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + ax\sin(x)$. Show that
$$|a| < 1 \Rightarrow \lim_{x\to\pm\infty} f(x) = \pm\infty$$
15. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the *zero set* of f
$$Z(f) = \{x; f(x) = 0\}$$
is a closed set. Conclude that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous then the zero set $\{x; f(x) = g(x)\}$ is closed.
16. Let $f : X \to \mathbb{R}$ be continuous. Show that for every $k \in \mathbb{R}$, the set of all $x \in X$ such that $f(x) \leq k$ is of the form $C \cap X$, where C is closed.
17. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the sets $\{x; f(x) < c\}$ and $\{x; f(x) > c\}$ are open for every $c \in \mathbb{R}$.
18. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the set $f^{-1}(A)$ is open for every open $A \subseteq \mathbb{R}$.
19. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ a closed set. Show that f is continuous if and only if the set $f^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}$.
20. Let $S \subseteq \mathbb{R}$ be nonempty. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by
$$f(x) = \inf\{|x - s|; s \in S\}$$
Show that f is *Lipschitz*: $\forall x, y \in \mathbb{R} \Rightarrow |f(x) - f(y)| \leq |x - y|$.
21. Let $X \subseteq \mathbb{R}$ be a closed set and $f : X \to \mathbb{R}$ continuous. Show that there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g_{|X} = f$.
22. Give an example of a bijective function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every $a \in \mathbb{R}$.
23. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every rational number to an irrational number, and vice-versa.
24. Let A be the set of all nonnegative algebraic numbers, and B be the set of negative transcendental numbers. Let $f : A \cup B \to [0, +\infty)$ be a function defined by $f(x) = x^2$. Show that f is a continuous bijection, whose inverse f^{-1} is discontinuous at every point, except zero.
25. (Brouwer Fixed Point Theorem) Let $f : [a, b] \to [a, b]$ be a continuous function. Show that there exists a point $x \in [a, b]$ such that $f(x) = x$. *[We call such point a 'fixed point'.]*

26. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If for every open set $A \subseteq \mathbb{R}$, the image $f(A)$ is open, then f is injective, hence monotone.
27. Fix $X \subseteq \mathbb{R}$. If every continuous function defined on X is bounded then X is compact.
28. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $\lim_{x \to -\infty} f(x) = \lim_{x \to +\infty} f(x) = +\infty$. Then f achieves its minimum value, i.e. there is $a \in \mathbb{R}$ such that $f(a) \leq f(x), \forall x \in \mathbb{R}$.
29. Show that $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-|x|}$ is a homeomorphism.
30. Classify all intervals of $\mathbb{R}$ up to homeomorphism. For example, all open intervals, whether or not bounded, are homeomorphic, hence should represent the same object.
31. Show that the inverse of f given in exercise 15, is uniformly continuous. (Notice that f isn't)
32. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ is uniformly continuous, but $g(x) = \sin x^2$ isn't.
33. Show that a polynomial $p : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if and only if has degree at most one.
34. Show that $f(x) = x^n$ is Lipschitz in any bounded set. Moreover, prove that if $n > 1$ and f is defined on an unbounded interval, then f is not even uniformly continuous.
35. Give an example of sets A, B open and a continuous function $f : A \cup B \to \mathbb{R}$ such that $f_{|A}, f_{|B}$ are uniformly continuous but f is not.
36. Given a function $f : X \to \mathbb{R}$. Suppose that for every $\epsilon > 0$, there exists $g : X \to \mathbb{R}$ continuous, such that $\forall x \in X$, $|f(x) - g(x)| < \epsilon$. Show that f is continuous.

Chapter 6
Derivatives

6.1 Definition and first properties

Let $X \subseteq \mathbb{R}$, $a \in X \cap X'$, and $f : X \to \mathbb{R}$ be a real valued function. We say f is *differentiable* at $a \in X$ if the following limit exists:

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{6.1}$$

The number $f'(a)$ is called the derivative of f at a. If f is differentiable at every $a \in X$, we simply say f is differentiable (in X).

Intuitively speaking, for $x \neq a$, the number $\frac{f(x)-f(a)}{x-a}$ is the slope of the secant line connecting the points $(x, f(x))$ and $(a, f(a))$, hence when $x \to a$, this number becomes the slope of the tangent line.

Similarly to one-sided limits, we can define *one-sided derivativesderivative!one-sided*, $f'_+(a) := \lim_{x \to a^+} \frac{f(x)-f(a)}{x-a}$, if $a \in X \cap X'_+$, and $f'_-(a) := \lim_{x \to a^-} \frac{f(x)-f(a)}{x-a}$ if $X \cap X'_-$. We can easily see that $f'(a)$ exists for some $a \in X \cap X'_+ \cap X'_-$ if and only if $f'_+(a)$ and $f'_-(a)$ exist and $f'_-(a) = f'_+(a)$. In particular, a function is not differentiable if its graph has sharp corners, since this implies $f'_-(a) \neq f'_+(a)$ at the corner.

If we set $h := x - a$ in equation 6.1, then we can see that $f'(a)$ can be equivalently defined by

$$f'(a) := \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}. \tag{6.2}$$

Sometimes the latter definition is more convenient for computational purposes.

If $a \in X'_+$ but $a \notin X'_-$, and $f'_+(a)$ exists, we can set $f'(a) := f'_+(a)$ and consider f to be differentiable at a. A similar convention holds for $a \in X'_-$. According to this convention, the function $f : [a, b) \to [a, b)$, given by $f(x) = x$, is differentiable.

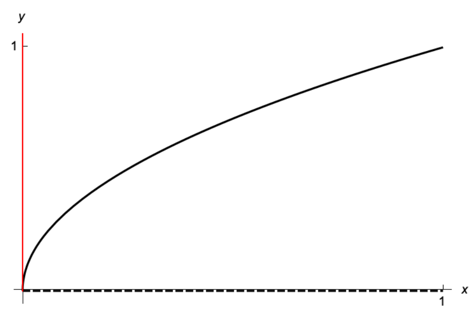
Example 6.1. Let $f : \mathbb{R} \to \mathbb{R}$ be linear, $f(x) = mx + b$. Then $f'(x) = m$. In particular, if $m = 0$ and $f(x) = b$ is constant, then $f'(x) = 0$.

Example 6.2. Consider $f(x) = |x|$. Using the definition of one-sided derivatives we obtain $f'_+(0) = 1$ and $f'_-(0) = -1$. Therefore, f is not differentiable at 0. On the other hand, we easily see that $f'(x) = 1$, if $x > 0$, and $f'(x) = -1$, if $x < 0$.

Example 6.3. Let $f : [0, +\infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Using equation 6.2, for $x > 0$, we obtain:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{h}{h(\sqrt{x+h} + \sqrt{x})} = \frac{1}{2x}$$

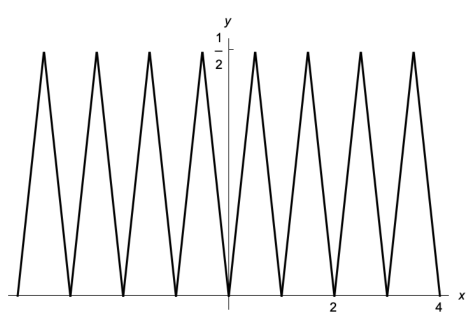
On the other hand, at $x = 0$ the quotient $\frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}} \to +\infty$ as $h \to 0^+$, hence $f'(0)$ doesn't exits. Intuitively, this is clear since the tangent line being a vertical line has ‘infinite’ slope.



Example 6.4. (Sawtooth function)Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

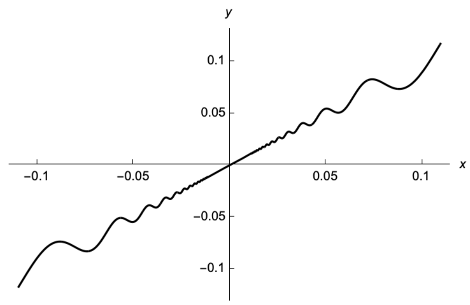
$$f(x) = \inf\{|x - n|; n \in \mathbb{Z}\}$$

.

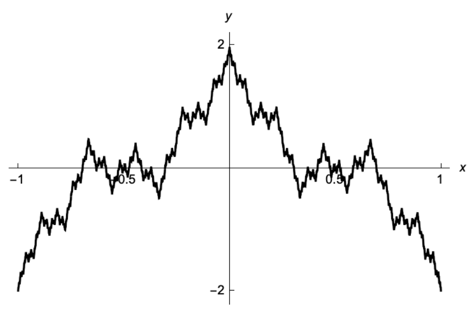


Notice that the graph of f has sharp corners at every $n, \frac{n}{2}$, for $n \in \mathbb{Z}$, hence it's not differentiable at those points. Otherwise, the function is differentiable with $f'(x) = \pm 1$, depending whether or not the fractional part of $f(x)$ is less than 0.5.

Example 6.5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x + 2x^2 \sin(1/x)$, if $x \neq 0$. Despite this seemly complicated definition, this function is indeed differentiable everywhere and $f'(x) = 1 - 2\cos(1/x) + 4x\sin(1/x)$



Example 6.6. (Weierstrass function) Given $0 < a < 1$ and $b \in \mathbb{N}$, such that $ab > 1 + \frac{3}{2}\pi$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sum_{n=1}^{\infty} a^n \cos(b^n \pi x)$. The figure below is the graph of $f(x)$. It is an example of a continuous function that is nowhere differentiable.



Moreover, the graph of $f(x)$ is self-similar if we zoom in, in the sense, that if we restrict the the domain of $f(x)$ to $[-\frac{1}{n}, \frac{1}{n}]$ and take n bigger and bigger, the shape of the graph doesn't change. We will prove these claims later, when we discuss series of functions.

Theorem 6.7. *A real valued function $f : X \to \mathbb{R}$ is differentiable at $a \in X$ if and only if there is number $C \in \mathbb{R}$ and a real valued function $r(x)$, such that if $a + h \in X$:*

$$f(a+h) = f(a) + Ch + r(h), \tag{6.3}$$

and $r(x)$ satisfies $\lim_{h \to 0} \frac{r(h)}{h} = 0$. Moreover, $C = f'(a)$.

Proof. The implication is clear. We prove the converse. Suppose that there is $C \in \mathbb{R}$ satisfying (6.3). Then

$$f(a+h) - f(a) - r(h) = Ch \tag{6.4}$$

Dividing both sides by h and taking the limit when $h \to 0$ we obtain

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=C\in\mathbb{R},$$

as required. □

The theorem above says that f is differentiable at a if and only if in a neighborhood of a, f can be approximated by the linear function $p(x)=f'(a)x+f(a)$ with error $r(x)$ that goes to zero faster than $g(x)=x$. We will see soon that the more derivatives f has, the better we can make this approximation using a polynomial $p(x)$ whose degree is equal to the number of derivatives of f.

If $f:X\to\mathbb{R}$ differentiable at $a\in X\cap X'$, we define the *differential at* a, denoted by $df_a:\mathbb{R}\to\mathbb{R}$, as the linear transformation given by

$$df_a(h)=f'(a)h. \tag{6.5}$$

In this notation, equation 6.3 becomes

$$f(a+h)=f(a)+df_a(h)+r(h). \tag{6.6}$$

Theorem 6.8. *If the $f:X\to\mathbb{R}$ is differentiable at $a\in X$ then f is continuous at $a\in X$.*

Proof. Indeed, we have

$$\begin{aligned}\lim_{x\to a}[f(x)-f(a)]=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)\right]&=\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\right]\cdot\lim_{x\to a}(x-a)\\&=f'(a)\cdot 0=0.\end{aligned} \tag{6.7}$$

$\therefore$ f is continuous at a. □

The theorem below follows directly from the definition of derivative and the properties of limits we have already proved.

Theorem 6.9. *(Properties of derivatives) If $f,g:X\to\mathbb{R}$ are differentiable at $a\in X\cap X'$ then $f\pm g$, $f\cdot g$, f/g (if $g'(a)\neq 0$) are also differentiable at a. Moreover,*

$$\begin{aligned}(f\pm g)'(a)&=f'(a)\pm g'(a)\\(f\cdot g)'(a)&=f'(a)\cdot g(a)+f(a)\cdot g'(a)\\\left(\frac{f}{g}\right)'(a)&=\frac{f'(a)g(a)-f(a)g'(a)}{g(a)^2}.\end{aligned} \tag{6.8}$$

Theorem 6.10. *(The Chain Rule) Let $f:X\to\mathbb{R}$ and $g:Y\to\mathbb{R}$ be real valued functions, such that $f(X)\subseteq Y$. If f is differentiable at $a\in X$, and g is differentiable at $b:=f(a)$, then $g\circ f:X\to\mathbb{R}$ is differentiable at a, moreover $(g\circ f)'(a)=g'(b)f'(a)$.*

Proof. By hypothesis, we have

$$
\begin{aligned}
(g \circ f)(a+h) = g[f(a+h)] &= g[f(a) + f'(a)h + r(h)] \\
&= g[f(a)] + g'[f(a)][f'(a)h + r(h)] + s(f'(a)h + r(h)) \\
&= g(b) + g'(b)[f'(a)h] + g'(b)[r(h)] + s(f(a+h) - f(a)).
\end{aligned}
$$

Since

$$\lim_{h\to 0} \frac{g'(b)[r(h)] + s(f(a+h) - f(a))}{h} = g'(b) \lim_{h\to 0} \frac{r(h)}{h} + \lim_{h\to 0} \frac{s(f(a+h) - f(a))}{h} = 0$$

The proof is complete by theorem 6.7. □

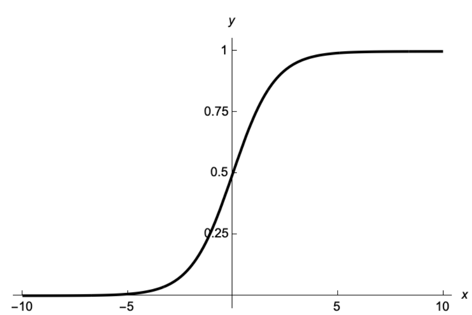
Corollary 6.11. *Let $f : X \to Y \subseteq \mathbb{R}$ be a bijective real valued functions. If f is differentiable at $a \in X$, and $f^{-1} : Y \to X$ is continuous at $b := f(a)$, then f^{-1} is differentiable at b if and only if $f'(a) \neq 0$, moreover, if that's the case, then $(f^{-1})'(b) = \frac{1}{f'(a)}$.*

Proof. If f^{-1} is differentiable at b, we can apply the Chain rule to $1 = (f^{-1} \circ f)'(a) = (f^{-1})'(b)f'(a)$. Conversely, suppose $f'(a) \neq 0$, set $g(y) := f^{-1}(y)$. Then

$$\lim_{y\to b} \frac{g(y) - g(b)}{y - b} = \lim_{y\to b} \frac{g(y) - a}{f[g(y)] - f(a)} = \lim_{y\to b} \left(\frac{f[g(y)] - f(a)}{g(y) - a} \right)^{-1} = \frac{1}{f'(a)} \tag{6.9}$$

$\therefore \; g'(b) = \frac{1}{f'(a)}$ and the theorem is proved. □

Example 6.12. (The Sigmoid function) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{1+e^{-x}}$, whose graph is shown below.



Using the chain rule, we have that

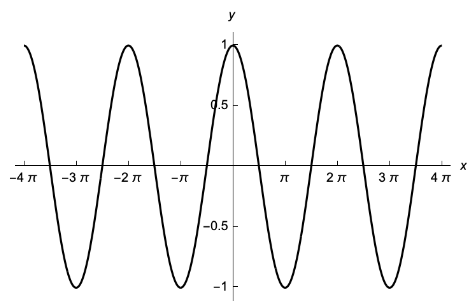
$$f'(x) = -\frac{1}{(1+e^{-x})^2}(-e^{-x}) = \frac{e^{-x}}{(1+e^{-x})^2}$$

6.2 Maximum and minimum points

The derivative of $f : X \longrightarrow \mathbb{R}$ at point $a \in X$ tells us crucial information about the behavior of the function in a neighborhood of a.

Let $f : X \longrightarrow \mathbb{R}$ be a real valued function and $a \in X$. We say f has a *local maximum* at a if there exists $\delta > 0$, such that $x \in (a - \delta, a + \delta) \Rightarrow f(x) \leq f(a)$. If the strict inequality $f(x) < f(a)$ is true, then a is called *strict local maximum*. Similar definitions are given to *local minimum* and *strict local minimum*.

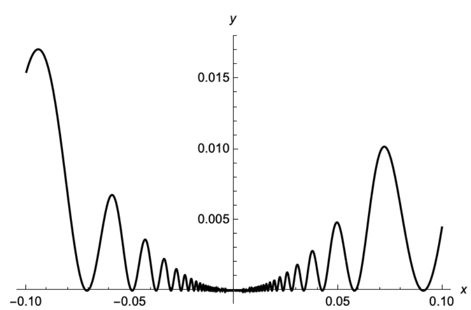
Example 6.13. The function $\cos : \mathbb{R} \longrightarrow \mathbb{R}$ has (strict) local maxima at points of the form $a = 2\pi n,\ n \in \mathbb{Z}$.



Similarly, $\cos x$ has (strict) local minima at points of the form $(2n - 1)\pi,\ n \in \mathbb{Z}$.

Example 6.14. The constant function given by $f(x) = C$ has (non-strict) local maxima and minima at every point of its domain.

Example 6.15. Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(0) = 0$ and $f(x) = x^2(1 + \sin \frac{1}{x})$, whose graph is shown below.



By definition, $f(x) \geq 0,\ \forall x \in \mathbb{R}$. Moreover, any neighborhood of 0 contains points whose image is 0. Hence, the point 0 is a (non-strict) local minimum.

Theorem 6.16. *Let $f : X \longrightarrow \mathbb{R}$ be differentiable from the right at $a \in X \cap X'_+$, i.e. $f'_+(a)$ exists. If $f'_+(a) > 0$ then we can find $\delta > 0$ such that $x \in (a, a + \delta) \Rightarrow f(x) > f(a)$. Similarly, if $f'_+(a) < 0$ then $\exists \delta > 0 : x \in (a, a + \delta) \Rightarrow f(x) < f(a)$.*

Proof. Follows directly from Corollary 5.6. □

A similar result is valid in the case $f'_-(a) > 0$ or $f'_-(a) < 0$.

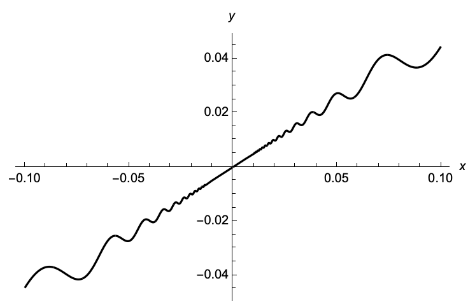
Corollary 6.17. *Let $f : X \longrightarrow \mathbb{R}$ be differentiable at $a \in X \cap X'_+ \cap X'_-$. If $f'(a) > 0$ then we can find $\delta > 0$ such that for all $x, y \in X$, we have $a - \delta < x < a < y < a + \delta \Rightarrow f(x) < f(a) < f(y)$.*

Notice that the corollary above *is not* saying that f is locally increasing.

Corollary 6.18. *Let $f : X \longrightarrow \mathbb{R}$ be differentiable at $a \in X \cap X'_+ \cap X'_-$. If f has a local maximum or minimum at $a \in X$ then $f'(a) = 0$.*

Example 6.19. The converse of Corollary 6.18 is false. The function $f(x) = x^3$ and $a = 0$ gives a counter-example.

Example 6.20. Consider the continuous function $f(x) = x^2 \sin \frac{1}{x} + \frac{x}{2}$ if $x \neq 0$ and $f(0) = 0$.



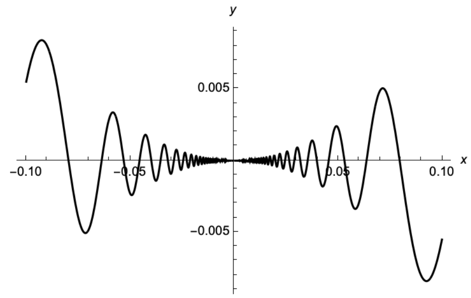
We have $f'(0) = \frac{1}{2} > 0$, but f is not increasing in any neighborhood I of 0. Indeed, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x} + \frac{1}{2}$, so we can pick $x \in I$ sufficiently small such that $\sin \frac{1}{x} = 0$ and $\cos \frac{1}{x} = 1$, for this $x \in I$ we have $f'(x) = -\frac{1}{2} < 0$, so f can't be increasing in I.

6.3 Derivative as a function

Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function defined on a interval I. We associate to f its derivative function $f' : I \longrightarrow \mathbb{R}$, whose value at each $x \in I$ is $f'(x)$.

When f' is continuous, we say f is *continuously differentiable*. The set of all continuously differentiable functions on a interval I is denoted by $C^1(I)$. In case $I = (-\infty, +\infty)$, we simply write $f \in C^1$ and say f is of class C^1.

Example 6.21. The function defined by $f(x) = x^2 \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$ is differentiable but $f \notin C^1$.



At $x = 0$ we have $f'(0) = 0$. However, $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ and $\lim_{x\to 0} f'(x)$ doesn't exists. Therefore, f' is not continuous at 0.

If $f : I \to \mathbb{R}$ is of class C^1, then we can apply the Intermediate Value Theorem to f' to conclude that: Given $a, b \in I$ such that $f'(a) < y < f'(b)$ for some $y \in \mathbb{R}$, then there exists $c \in I$ such that $y = f'(c)$.

The following theorem strengthens the above by removing the continuity assumption of f'.

Theorem 6.22. *(Darboux's theorem) Let $f : [a, b] \to \mathbb{R}$ be differentiable. If $f'(a) < y < f'(b)$, then there exists $c \in I$ such that $y = f'(c)$.*

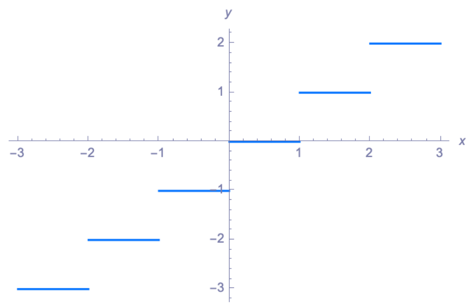
Proof. It suffices to prove the result when $y = 0$ and then consider $g(x) = f(x) - yx$. From the fact that $f'(a) < 0 < f'(b)$, we know that $f(x) < f(a)$ in a neighborhood of a, and $f(x) < f(b)$ in a neighborhood of b. That implies that f achieves its minimum (see corollary 5.60) at a point $c \in (a, b)$, by 6.18 we must have $f'(c) = 0$. □

Example 6.23. The corollary above says that the Dirichlet function $f(x) = 1$, if $x \in \mathbb{Q} \cap [0, 1]$, $f(x) = 0$ if $x \in (\mathbb{R} - \mathbb{Q}) \cap [0, 1]$ can't be the derivative of a function defined on $[0, 1]$.

Corollary 6.24. *Let $f : I \to \mathbb{R}$ be a differentiable function on an interval I. Then f' doesn't have jump discontinuities.*

Proof. We claim that given a point $a \in I$, if the one sided limits $\lim_{x\to a^+} f'(x)$, $\lim_{x\to a^-} f'(x)$ exist, then $f'(x)$ is continuous at a. Suppose $R = \lim_{x\to a^+} f'(x)$ exists but $R \neq f'(a)$, say $R > f'(a)$. Take $y \in \mathbb{R}$ such that $f'(a) < y < R$. Then there exists $\delta > 0$ such that $x \in (a, a + \delta) \Rightarrow f'(x) > y$. In particular, $f'(a) < R < f'(a + \frac{\delta}{2})$ but there is no $c \in (a, a + \frac{\delta}{2})$ such that $f'(c) = R$, a contradiction. Using a similar argument, we conclude the equivalent result if $\lim_{x\to a^-} f'(x)$ exists. □

Example 6.25. The corollary above says that the floor function $f(x) = \lfloor x \rfloor$, can't be the derivative of a function defined on $\mathbb{R}$.



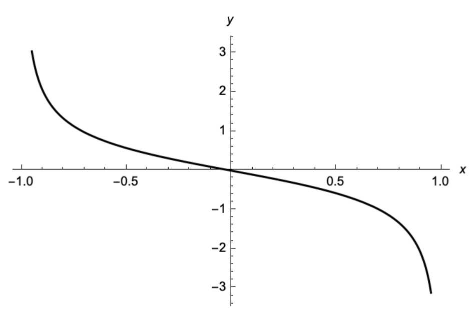
Theorem 6.26. *(Rolle) Let $f : [a,b] \to \mathbb{R}$ be continuous satisfying $f(a) = f(b)$. If f is differentiable on (a,b) then there exists $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. If f is constant then $f'(x) = 0$, so suppose f not constant. Since f is continuous on $[a,b]$, it achieves its maximum and minimum in $[a,b]$. Since $f(a) = f(b)$, the maximum/minimum can't be at an endpoint, otherwise the function would be constant. Hence, the function has at least one maximum or minimum in the interior (a,b), at that point the derivative must be zero by Corollary 6.18. □

Notice that we didn't use $f'(a)$ or $f'(b)$ in the proof, hence the requirement that f be differentiable in (a,b) and not in $[a,b]$.

Example 6.27. The absolute value function $f(x) = |x|$ when defined on $[-1,1]$ is continuous and satisfies $f(-1) = f(1)$, but there is no point $c \in [-1,1]$ such that $f'(c) = 0$. This is not a counter-example to Theorem 6.26, because f is not differentiable at $0 \in [-1,1]$.

Example 6.28. The function $f(x) = \sqrt{1-x^2}$ is continuous on $[0,1]$ but it's differentiable only in $(0,1)$, since it's derivative $f'(x) = -\frac{x}{\sqrt{1-x^2}}$ is discontinuous at ± 1, as the picture below suggests.

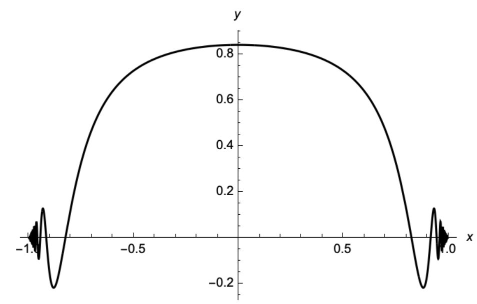


Still, Rolle's theorem guarantees the existence of a point $c \in [0,1]$ with $f'(c) = 0$. Indeed, $c = 0$ in this case.

Example 6.29. (The headphone function) The function $f : [-1,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0, \text{ if } |x| = 1 \\ (1 - x^2) \sin \frac{1}{1-x^2}, \text{ if } |x| \neq 1 \end{cases}$$

is another example of function continuous on $[-1, 1]$ but differentiable only in $(-1, 1)$.



Theorem 6.30. *(Lagrange's Mean Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous. If f is differentiable on (a, b) then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Set $g(x) = \frac{f(b)-f(a)}{b-a}(x - a) + f(a)$. Then g satisfies $g(a) = f(a)$ and $g(b) = f(b)$. If we set $h(x) = f(x) - g(x)$, the function h satisfies $h(a) = h(b)$, hence by Rolle's theorem $h'(c) = 0$ for some $c \in (a, b)$. The result follows. □

Corollary 6.31. *Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f'(x) = 0$ for every $x \in (a, b)$. Then f is constant.*

Corollary 6.32. *Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions such that $f'(x) = g'(x)$ for every $x \in (a, b)$. Then $f(x) = g(x) + C$, for some constant $c \in \mathbb{R}$.*

Corollary 6.33. *Any function $f : I \to \mathbb{R}$ defined on a interval such that $x \in I \Rightarrow |f'(x)| \leq C$ for some $C \in \mathbb{R}$, is Lipschitz.*

Corollary 6.34. *Let $f : I \to \mathbb{R}$ be differentiable in an interval I. Then $f'(x) \geq 0$ if and only if f is nondecreasing in I. In case $f'(x) > 0$, then f is increasing. Equivalent statements are true if $f'(x) \leq 0$ and f nonincreasing.*

Proof. Suppose $f'(x) \geq 0$ and $x, y \in I$ such that $x \leq y$. By the Mean Value Theorem, $f(y) - f(x) = f'(c)(y - x) \geq 0$, and we conclude that $f(x) \leq f(y)$. Conversely, if f is nondecreasing then for every $x \in I$ such that $x + h \in I$, we have that the ratio $\frac{f(x+h)-f(x)}{h}$ is always nonnegative, hence its limit when $h \to 0$ is also nonnegative. The same argument *mutatis mutandis* applies in the strict inequality. □

Example 6.35. As a nice application of the Mean Value theorem we show that $\lim(\sqrt{n+1}-\sqrt{n}) = 0$. Consider the function $f : [n, n+1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$. Using the Mean Value Theorem we can find $c \in (n, n+1)$ such that

$$f'(c) = \frac{\sqrt{n+1}-\sqrt{n}}{(n+1)-n},$$

or equivalently

$$\sqrt{n+1}-\sqrt{n} = \frac{1}{2c} \leq \frac{1}{2n}.$$

Using the Squeeze theorem we conclude that $\lim(\sqrt{n+1}-\sqrt{n}) = 0$.

6.4 Taylor's Theorem

Let $f : I \to \mathbb{R}$ be a real valued function defined on an interval I. The n-th derivative of f, if exists, is defined inductively by setting $f''(x) = (f')'(x)$ and $f^{(n)}(x) = (f^{(n-1)})'(x)$ for $n \in \mathbb{N}$. By convention, we set $f^0(x) = f(x)$.

We say that f is of class C^k in I, denoted by $f \in C^k(I)$, if $f^{(k)}$ exists and is continuous in I. When $I = \mathbb{R}$, we simply write $f \in C^k$. Recall that $f \in C^0$, means f is continuous, so the definition makes sense even if k is zero.

In case $f \in C^k(I)$ for every $k \in \mathbb{N}$, we say that f is *smooth* and write $f \in C^\infty(I)$. Equivalently, a function f is smooth if $f^{(n)}$ exists for every $n \in \mathbb{N}$.

The following example generalizes example 6.21.

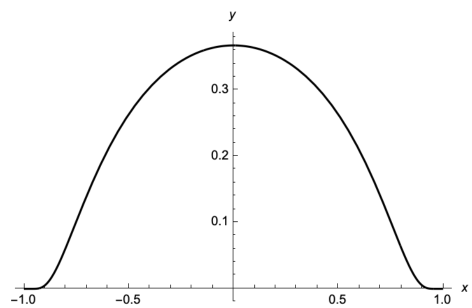
Example 6.36. The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|x$ is C^1 but it's not C^2. Indeed, we can easily check that its derivative is given by

$$f'(x) = \begin{cases} 2x, & x \geq 0 \\ -2x, & x < 0 \end{cases}$$

which is continuous everywhere. Whereas, f'' has a jump discontinuity at zero, so $f \notin C^2$. More generally, the function $g(x) = |x|x^n$ is in C^n but $g \notin C^{n+1}$.

Example 6.37. (Standard Mollifier) Consider the function defined by:

$$f(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$



We can easily see that $f \in C^\infty$ and the set where $f \neq 0$ is bounded, hence has compact closure. This type of function and its higher dimensional generalization are extensively used in the field of differential equations.

Example 6.38. Since $\sin' x = \cos x$ and $\cos' x = -\sin x$, we deduce that $\sin x, \cos x \in C^\infty$. Similarly, $e^x, \log x$ and any polynomial are examples of smooth functions.

Let $f : I \longrightarrow \mathbb{R}$ be a real valued function defined on an interval $I \subseteq \mathbb{R}$ having derivatives up to order n at $a \in I$, i.e. $f^{(n)}(a)$ exists. The polynomial $p(x)$ defined by

$$p(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (6.10)$$

is called the *Taylor polynomial* of order n of f at a.

Equivalently, the n-th order Taylor polynomial of f at a is the unique polynomial $p(x)$ of degree n, such that $f^{(k)}(a) = p^{(k)}(a)$ for $k = 1, 2, \dots, n$.

Theorem 6.39. *(Taylor's Theorem) Let $f : I \longrightarrow \mathbb{R}$ be a real valued function having derivatives up to order n at $a \in I$, and $p(x)$ be the n-th order Taylor polynomial at a. Then the function $r : I \longrightarrow \mathbb{R}$, defined by $r(x) = f(x) - p(x)$, i.e.*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + r(x),$$

satisfies $\lim\limits_{x \to a} \frac{r(x)}{(x-a)^n} = 0.$

Proof. Recall that the case $n = 1$ was proved in theorem 6.7. Suppose $n = 2$, we use the Mean Value Theorem to obtain c between x and a such that:

$$\frac{r(x)}{(x-a)^2} = \frac{r(x) - r(a)}{(x-a)^2} = \frac{r'(c)(x-a)}{(x-a)^2} = \frac{r'(c)}{x-a} = \frac{[r'(c) - r'(a)](c-a)}{(c-a)(x-a)}$$

$\therefore \lim\limits_{x \to a} \frac{r(x)}{(x-a)^2} = 0$, since $r^{(2)}(a) = 0$ and $\left|\frac{c-a}{x-a}\right| \leq 1$. Using the same argument, we can prove the result for any value n. □

Corollary 6.40. *(L'Hôpital's rule) Let $f, g : I \to \mathbb{R}$ be real valued functions having derivatives up to order n at $a \in I$, such that $f^{(k)}(a) = g^{(k)}(a) = 0$, for $k = 0, 1, 2, \ldots, n-1$, but $f^{(n)}(a)$ and $g^{(n)}(a)$ are non-zero. Then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}.$$

Proof. By Taylor's formula and the hypothesis of the corollary, we have:

$$\frac{f(x)}{g(x)} = \frac{\frac{f^{(n)}(a)}{n!} + \frac{r(x)}{(x-a)^n}}{\frac{g^{(n)}(a)}{n!} + \frac{s(x)}{(x-a)^n}},$$

for some $r(x), s(x)$, satisfying $\frac{r(x)}{(x-a)^n} \to 0$ and $\frac{s(x)}{(x-a)^n} \to 0$, when $x \to a$. The corollary is then immediate. □

Corollary 6.41. *Let $f : I \to \mathbb{R}$ be real valued function having derivative up to order n at $a \in \operatorname{int}(I)$, such that $f^{(k)}(a) = 0$, for $k = 1, 2, \ldots, n-1$, but $f^{(n)}(a) \neq 0$. Then if n is odd, the point a is not a local maximum or minimum, and if n is even, two outcomes are possible: $f^{(n)}(a) > 0$ implies the point a is a strict local minimum; $f^{(n)}(a) < 0$ implies the point a is a strict local maximum.*

Proof. Notice that in this case Taylor's formula can be written as

$$f(a+h) - f(a) = h^n \left[\frac{f^{(n)}(a)}{n!} + \frac{r(a+h)}{h^n} \right]$$

for $h \in \mathbb{R}$ such that $a + h \in I$. Since $\frac{r(a+h)}{h^n} \to 0$ when $h \to 0$, for h sufficiently small, say $0 < |h| < \delta$, the expression in the square brackets has the same sign as $f^{(n)}(a)$. Hence, if n is odd, we can always find $h_1, h_2 \in I$ such that $f(a+h_1) - f(a) > 0$ and $f(a+h_2) - f(a) < 0$, so a can't be a local maximum or minimum.

Now, suppose n is even. Then if $f^{(n)}(a) > 0$, the above discussion implies $f(a+h) - f(a) > 0$ for $0 < |h| < \delta$, hence a is a local minimum. Similarly, if $f^{(n)}(a) < 0$ we must have $f(a+h) - f(a) < 0$, and a is a local maximum. □

We can enhance Taylor's Theorem if we require f to be of Class C^n and having the $f^{(n+1)}$ derivative, instead of just having the f^n derivative, which is not necessarily continuous.

Theorem 6.42. *(Taylor's Theorem with Lagrange Remainder) Let $f : [a, b] \to \mathbb{R}$ be a real valued function such that $f \in C^n$ and $f^{(n+1)}(x)$ exists in (a, b). Then there exists $c \in (a, b)$ such that*

$$f(b) = f(a) + f'(a)(b-a) + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(b) - f(x) - f'(x)(b-x) + \ldots + \frac{f^{(n)}(x)}{n!}(b-x)^n + \frac{C}{(n+1)!}(b-x)^{n+1},$$

where C is the unique number such that $g(a) = 0$.

The function g is continuous on $[a, b]$, differentiable in (a, b), and satisfies $g(a) = g(b)$. Therefore, by Rolle's Theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. On the other hand, a quick computation gives:

$$g'(x) = \frac{C - f^{(n+1)}(x)}{n!}(b-x)^n,$$

We conclude that $C = f^{(n+1)}(c)$, and the theorem becomes the statement $g(a) = 0$. □

Let $f : I \to \mathbb{R}$ be a smooth function, i.e. $f \in C^\infty$, and $a \in I^\circ$. Using Taylor's Theorem with Lagrange remainder, for each $n \in \mathbb{N}$ we have:

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1} + r_n(x), \qquad (6.11)$$

where $r_n(x) = \frac{f^{(n)}(c)}{n!}(x-a)^n$ and c is between x and a. It is then natural to ask what happens when we let $n \to +\infty$ in (6.11).

The series $f(a)+f'(a)(x-a)+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$, is called the *Taylor Series* of f at $a \in I$. Notice that it's not entirely clear that the Taylor Series of f at a has to coincide with $f(x)$, in fact, it's possible for the Taylor Series to diverge and even if it converges, it could converge to a number other than $f(x)$.

A function $f : I \to \mathbb{R}$ is called *Analytic* if for every $a \in I$, there exists $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n,$$

In other words, a function is analytic if it coincides with its Taylor series in a neighborhood of every point of its domain. Notice that it follows from (6.11) that a function is analytic if and only if for every $x \in I$, we have $\lim_{n\to\infty} r_n(x) = 0$.

Example 6.43. Any polynomial $p(x)$ is clearly analytic, since $p^{(n)}(x)$ vanishes for sufficiently large $n \in \mathbb{N}$.

Example 6.44. The exponential function $f(x) = e^x$ is perhaps one of the most famous analytic functions. We use Taylor's theorem (with $a = 0$), to obtain:

$$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + e^{c_n}\frac{x^n}{n!}$$

with $|c_n| < |x|$. Since $\lim \frac{x^n}{n!} = 0$, the Taylor series for e^x at 0 converges to e^x. Moreover, notice that $e^{x+a} = e^x e^a$, hence the Taylor series for e^x converges at any point $a \in \mathbb{R}$, and e^x is analytic.

Example 6.45. Let $x \in \mathbb{R}$, then

$$1 + x + x^2 + \ldots + x^{n-1} + \frac{x^n}{1-x} = \frac{1}{1-x}.$$

Consider the function $f : (0,1) \to \mathbb{R}$ given by $f(x) = \frac{1}{1-x}$. Then using Taylor's Theorem we obtain $r_n(x) = \frac{x^n}{1-x}$ in this case, so $\lim_{n\to\infty} r_n(x) = 0$, which implies $f(x) = \sum_{n=0}^{\infty} x^n$. Hence, $f(x)$ agrees with its Taylor Series at 0.

Example 6.46. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \cos x$. Using Taylor's theorem around the origin (with $a = 0$), we can write

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + r_{2n+1}(x)$$

where $r_n(x) = [\cos x^{(n)}](c)\frac{x^n}{n!}$. Notice that

$$0 \leq |r_n(x)| \leq \frac{|x|^{2n+1}}{(2n+1)!},$$

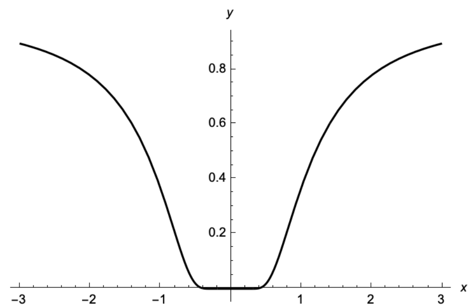
and recall that by example 3.53, $\lim_{n\to\infty} \frac{|x|^{2n+1}}{(2n+1)!} = 0$. We conclude that $\lim_{n\to\infty} r_n(x) = 0$ and it follows that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + \ldots.$$

Hence, the Taylor series of $\cos x$ at 0 converges to $\cos x$ at every point $x \in \mathbb{R}$. The same argument can be applied if if the Taylor series is not centered at zero ($a \neq 0$). In conclusion, the function $\cos x$ is analytic.

Example 6.47. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, \ x \neq 0 \\ 0, \ x = 0 \end{cases}$$



Using the fact that $\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$ for any $n \geq 0$, we can see that $f^{(n)}(0) = 0$, and the function f is smooth. However, the Taylor series at 0 is identically zero, since $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 0$. In particular, since $x \neq 0 \Rightarrow f(x) \neq 0$, it's impossible for $f(x)$ to be analytic on $\mathbb{R}$.

Exercises

1. Let $f, g, h : X \longrightarrow \mathbb{R}$ be functions such that, for every $x \in X$ we have $f(x) \leq g(x) \leq h(x)$. Show that if there is a point $a \in X \cap X'$ such that $f(a) = h(a)$ and $f'(a) = h'(a)$ then $g'(a)$ exists and $g'(a) = f'(a) = h'(a)$.
2. Let $p : \mathbb{R} \longrightarrow \mathbb{R}$ be an odd degree polynomial. Then there exists $c \in \mathbb{R}$ such that $p''(c) = 0$.
3. Let $f : X \longrightarrow \mathbb{R}$ be differentiable at $a \in X \cap X'$. If x_n and y_n are sequences in X such that $\lim x_n = \lim y_n = a$ and $x_n < a < y_n$ for every $n \in \mathbb{N}$, show that
$$\lim \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$
4. Show that the function given by $f(0) = 0$, $f(x) = x^2 \sin \frac{1}{x}$ if $x \neq 0$, is differentiable. Find sequences x_n and y_n such that $x_n \neq y_n$, $\lim x_n = \lim y_n = 0$ but $\lim \frac{f(y_n) - f(x_n)}{y_n - x_n}$ doesn't exist.
5. Let $f : I \longrightarrow \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. We call $a \in I$ a *critical point* if $f'(a) = 0$. We say a critical point a is *non-degenerate* if $f''(a) \neq 0$.

    a) If $f \in C^1$, show that the set of all critical points contained in a closed interval $[c, d] \subseteq I$ is closed.
    b) Show that local maximum and minimum points of f are critical points. Moreover, any critical non-degenerate point is a maximum or minimum.
    c) Show that there are C^∞ functions with isolated degenerate local maximum/minimums. Moreover, there are critical points of C^∞ functions that are not local maximum/minimum points.
    d) Show that every non-degenerate critical point of f is isolated.
    e) Let $f \in C^1$, suppose that the critical points of f contained in a closed interval $[c, d] \subseteq I$ are non-degenerate. Show that there are finitely many of them. Conclude that f has at most a countable number of non-degenerate critical points in I.
    f) The function $f(0) = 0$, $f(x) = x^4 \sin \frac{1}{x}$ if $x \neq 0$ has infinitely many non-degenerate critical points in $[0, 1]$. Wouldn't this be a contradiction to 5.4? Why/why not?

6. Let $f : I \longrightarrow \mathbb{R}$ be a function defined on interval $I \subseteq \mathbb{R}$. If there is $C, \alpha > 0$ such that $\forall x, y \in I \Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$, we say f is *Holder continuous*. Show that if $\alpha > 1$ then f is constant.
7. Let $f : I \longrightarrow \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. Show that if $f'(x) = 0$ for every $x \in I$ then f is constant.
8. Show that a differentiable function $f : I \longrightarrow \mathbb{R}$ is Lipschitz, i.e. $|f(x) - f(y)| \leq C|x - y|$, if and only if $|f'(x)| \leq C$.
9. Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f \in C^\infty$, $f(x) \neq x$, $\forall x \in \mathbb{R}$ and $|f'(x)| < 1$.
10. Let $f : [0, \pi] \longrightarrow \mathbb{R}$ be defined by $f(x) = \cos(\cos(x))$. Show that $|f'(x)| \leq c < 1$ for some $c \in \mathbb{R}$.

11. Let $f : (a, +\infty) \to \mathbb{R}$ be differentiable. Show that if $\lim_{x\to+\infty} f(x) = b$ and $\lim_{x\to+\infty} f'(x) = c$, then $c = 0$. *[Hint: Apply the Mean Value theorem on* $[n, n+1]$ *and let* $n \to +\infty$*.]*
12. Let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable on (a, b), satisfying $f(a) = f(b)$. Given $k \in \mathbb{R}$, show that $\exists c \in (a, b)$ such that $f'(c) = kf(c)$. *[Hint: Apply Rolle's theorem to* $g(x) = f(x)e^{-kx}$*.]*
13. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. A *root* of f is a number $c \in I$ such that $f(c) = 0$. Show that between two consecutives roots of f', there is at most one root of f.
14. Let $f : [0, +\infty) \to \mathbb{R}$ be twice differentiable. Show that if f'' is bounded and $\lim_{x\to+\infty} f(x)$ exists, then $\lim_{x\to+\infty} f'(x) = 0$.
15. Show that the composition of C^k functions is still a C^k function.
16. Given $a, b \in \mathbb{R}$ with $a < b$, consider $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} e^{\frac{1}{(x-a)(x-b)}}, & \text{if } x \in (a, b), \\ 0, & \text{if } x \notin (a, b). \end{cases}$$

Show that $\varphi \in C^\infty$ and φ has exactly one maximum point.
17. Let $f : I \to \mathbb{R}$ be twice differentiable at $a \in I^\circ$. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$

Given a example where the limit above exists but $f'(a)$ doesn't.
18. Show that the function $f(x) = |x|^{2n+1}$ is of class C^{2n} but $f^{(2n+1)}(x)$ doesn't exist in every $a \in \mathbb{R}$.

Chapter 7
Integrals

7.1 Integrable functions

Let $[a, b] \subseteq \mathbb{R}$ be a closed interval. A *partition* of $[a, b]$ is a finite subset $P = \{x_0, x_1, \ldots, x_n\} \subseteq [a, b]$, such that $x_0 = a$ and $x_n = b$.

By convention, the elements of a partition are written in increasing order:

$$P = \{a = x_0 < x_1 < x_2 < \ldots < x_n = b\}.$$

Let P, Q be partitions of $[a, b]$. We say that the partition Q is a *refinement* of the partition P if $P \subseteq Q$. More precisely, Q is obtained from P by adding a finite number of points.

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Set $m = \inf f$ and $M = \sup f$, then:

$$m \leq f(x) \leq M, \ \forall x \in [a, b].$$

If $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$, we denote

$$m_i := \inf\{f(x); x_{i-1} \leq x \leq x_i\} \text{ and } M_i := \sup\{f(x); x_{i-1} \leq x \leq x_i\},$$

and define the *oscillation* of f at $[x_{i-1}, x_i]$ by

$$\omega_i := M_i - m_i.$$

If f is continuous, the values m_i, M_i, ω_i are achieved by Weierstrass Extreme Value Theorem.

We define the *lower sum* of f with respect to P by

$$s(f; P) = m_1(x_1 - x_0) + \cdots + m_n(x_n - x_{n-1}) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}),$$

and likewise, the *upper sum* of f with respect to P by

$$S(f;P) = M_1(x_1 - x_0) + \cdots + M_n(x_n - x_{n-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

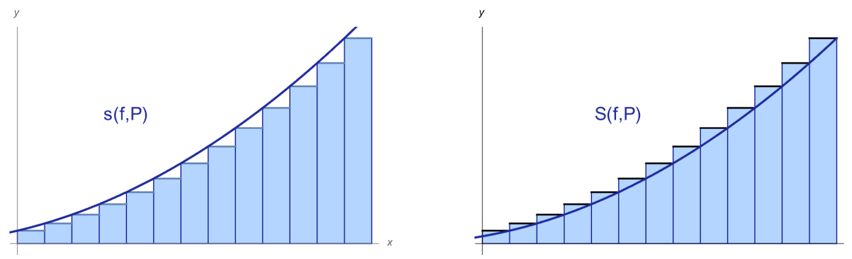


Fig. 7.1: Representation of $s(f;P)$ and $S(f;P)$

By definition, we have

$$m(b-a) \leq s(f;P) \leq S(f;P) \leq M(b-a) \text{ and } S(f;P) - s(f;P) = \sum_{i=1}^{n} \omega_i(x_i - x_{i-1}).$$

When $f \geq 0$, the number $s(f;P)$ represents an approximation of the area under the graph of f using rectangles that are below the graph, whereas $S(f;P)$ represents an approximation using rectangles above the graph of f.

Let $\mathcal{P} = \{P; P \text{ is a partition of } [a,b]\}$ and $f : [a,b] \to \mathbb{R}$ be a bounded function. The *lower integral* and *upper integral* are defined respectively by:

$$\underline{\int_a^b} f(x)\mathrm{dx} := \sup_{P \in \mathcal{P}} s(f;P) \text{ and } \overline{\int_a^b} f(x)\mathrm{dx} := \inf_{P \in \mathcal{P}} S(f;P),$$

Theorem 7.1. *Let $P, Q \in \mathcal{P}$. Then*

$$P \subseteq Q \Rightarrow s(f;P) \leq s(f;Q) \textit{ and } S(f;Q) \leq S(f;P)$$

Proof. It's enough to prove the result when $Q = P \cup \{a\}$. Suppose $P = \{x_0 < x_1 < \ldots < x_n\}$ and $x_{k-1} < a < x_k$ for some $k \leq n$. Define

$$m' := \inf_{x \in [x_{k-1}, a]} f(x) \text{ and } m'' := \inf_{x \in [a, x_k]} f(x).$$

Notice that m_k is less than or equal to m', m''. We have:

$$\begin{aligned} s(f;Q) - s(f;P) &= m'(a - x_{k-1}) + m''(x_k - a) - m_k(x_k - x_{k-1}) \\ &= (m'' - m_k)(x_k - a) + (m' - m_k)(a - x_{k-1}) \qquad (7.1) \\ &\geq 0 \end{aligned}$$

A similar argument shows that $S(f;Q) \leq S(f;P)$. $\square$

The figure below illustrates theorem 7.1 for a partition P and a refinement $Q \supseteq P$, when $f(x) = \frac{1}{x}$. The sum of the highlighted rectangles represent $s(f;P)$ and $s(f;Q)$ respectively. It's easy to see that $s(f;Q) \geq s(f;P)$.

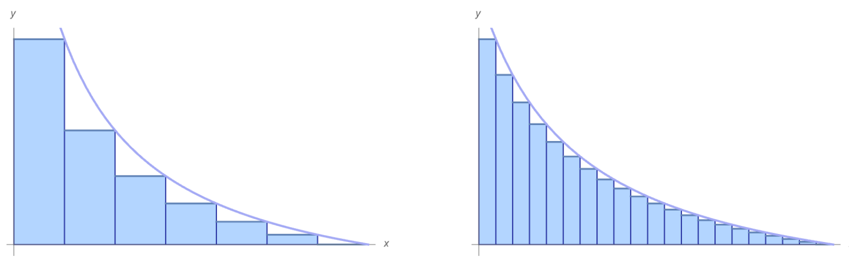


Fig. 7.2: Representation of $s(f;P)$ and $s(f;Q)$

Corollary 7.2. *For any partitions $P, Q \in \mathcal{P}$ we have*

$$s(f;P) \leq S(f;Q)$$

Proof. Apply Theorem 7.1 to P and $P \cup Q$ (Q and $P \cup Q$). $\square$

Lemma 7.3. *Let $X, Y \subseteq \mathbb{R}$ be sets satisfing*

$$x \leq y, \forall x \in X, \forall y \in Y,$$

then $\sup X \leq \inf Y$. *Moreover, the equality* $\sup X = \inf Y$ *holds if and only if given* $\epsilon > 0$, *there are $x \in X, y \in Y$ such that $y - x < \epsilon$.*

Proof. By definition, every $y \in Y$ is an upper bound for X hence $\sup X \leq y$, for every $y \in Y$. On the other hand, $\sup X$ is a lower bound for Y, thus $\sup X \leq \inf Y$. Suppose $\sup X = \inf Y$ and $\epsilon > 0$ is given. Then $\sup X - \frac{\epsilon}{2}$ is not an upper bound, so $\exists x \in X$ such that $\sup X - \frac{\epsilon}{2} < x \leq \sup X$. Similarly, we can find $y \in Y$ such that $\inf Y \leq y < \inf Y + \frac{\epsilon}{2}$. Therefore, $y - x < \inf Y + \frac{\epsilon}{2} - \sup X + \frac{\epsilon}{2} = \epsilon$. Conversely, suppose $\sup X < \inf Y$. If we set $\epsilon = \inf Y - \sup X$, then $y - x \geq \epsilon$. $\square$

Theorem 7.4. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, say $m \leq f(x) \leq M$, then:*

$$m(b-a) \leq \underline{\int_a^b} f(x)\mathrm{d}x \leq \overline{\int_a^b} f(x)\mathrm{d}x \leq M(b-a)$$

Proof. The proof of the middle inequality follows directly from lemma 7.3. The other two inequalities are obvious. $\square$

A bounded function $f : [a, b] \to \mathbb{R}$ is *(Riemann) integrable* if

$$\underline{\int_a^b} f(x)\mathrm{dx} = \overline{\int_a^b} f(x)\mathrm{dx},$$

and we denote this common value by $\int_a^b f(x)\mathrm{dx}$, or simply, by $\int_a^b f$.

Example 7.5. The constant function $f : [a, b] \to \mathbb{R}$ given by $f(x) = C$ is clearly integrable since $s(f; P) = S(f; P) = C(b - a)$ for any partition P.

Example 7.6. The Dirichlet function $f : [0, 1] \to \mathbb{R}$ given by $f(x) = 1$ if $x \in \mathbb{Q}$, and 0 otherwise, is not integrable since $s(f; P) = 0$ and $s(f; P) = b - a$ for any partition P.

Theorem 7.7. *(Cauchy criterion)Let $f : [a, b] \to \mathbb{R}$ be a bounded function. The following are equivalent:*

(1) f is integrable,
(2) For every $\epsilon > 0$, there are partitions P and Q of $[a, b]$ such that $S(f; Q) - s(f; P) < \epsilon$,
(3) For every $\epsilon > 0$, there is a partition $R = \{x_0 < x_1 < \ldots < x_n\}$ of $[a, b]$ such that $S(f; R) - s(f; R) = \sum\limits_{k=1}^{n} \omega_k (x_k - x_{k-1}) < \epsilon$.

Proof. The fact that (1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) follows directly from lemma 7.3. Suppose (2) is true and set $R = P \cup Q$, then

$$s(f; P) \le s(f; R) \le S(f; R) \le S(f; Q),$$

$\therefore\ S(f; R) - s(f; R) < \epsilon$, and (2) $\Rightarrow$ (3). □

Recall given a function $f : [a, b] \to \mathbb{R}$, the oscillation of f in I is $\omega(I) = \sup\limits_I f - \inf\limits_I f$. We define the *oscillation* of f around a point c by $\omega(f, c) := \lim\limits_{\delta \to 0} \omega(c - \delta, c + \delta)$.

Theorem 7.8. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is continuous at $c \in [a, b]$ if and only if $\omega(f, c) = 0$.*

Proof. Suppose f continuous at c. Then given $\epsilon > 0$ we can find $\delta > 0$ such that for every $x \in [a, b]$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2} \Rightarrow f(c) - \frac{\epsilon}{2} < f(x) < f(c) + \frac{\epsilon}{2}$, thus $\omega(c - \delta, c + \delta) < \epsilon$. Conversely, suppose $\omega(f, c) = 0$. Given $\epsilon > 0$, there exists $\delta > 0$ such that for $x, y \in [a, b]$, $x, y \in (c - \delta, c + \delta) \Rightarrow |f(x) - f(y)| < \epsilon$, in particular for $y = c$ we have $x \in (c - \delta, c + \delta) \Rightarrow |f(x) - f(c)| < \epsilon$, the conclusion follows. □

7.2 Properties of Integrals

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For simplicity, we adopt the following conventions:

$$\int_a^a f = 0 \text{ and } \int_b^a f = -\int_a^b f$$

Theorem 7.9. *Let $a < c < b$. Then $f : [a, b] \to \mathbb{R}$ is integrable if and only if $f_{|[a,c]}$ and $f_{|[c,b]}$ are integrable. In the affirmative case, we have*

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

Proof. Consider the sets

$$\begin{aligned} A &= \{s(f_{|[a,c]}; P); P \text{ is a partition of } [a, c]\}, \\ B &= \{s(f_{|[c,b]}; P); P \text{ is a partition of } [c, b]\}, \\ C &= \{s(f; P); P \text{ is a partition of } [a, b] \text{ and } c \in P\}. \end{aligned}$$

Notice that by Theorem 7.1, $\underline{\int}_a^b f = \sup C$. It follows that

$$\underline{\int}_a^b f = \sup(A + B) = \sup A + \sup B = \underline{\int}_a^c f + \underline{\int}_c^b f,$$

and similarly,

$$\overline{\int}_a^b f = \overline{\int}_a^c f + \overline{\int}_c^b f.$$

$$\therefore \quad \overline{\int}_a^b f - \underline{\int}_a^b f = \left(\overline{\int}_a^c f - \underline{\int}_a^c f\right) + \left(\overline{\int}_c^b f - \underline{\int}_c^b f\right).$$

We conclude that $\overline{\int}_a^b f = \underline{\int}_a^b f$ if and only if $\overline{\int}_a^c f = \underline{\int}_a^c f$ and $\overline{\int}_c^b f = \underline{\int}_c^b f$. □

Example 7.10. (Step functions) Given a set $X \subseteq \mathbb{R}$, consider the function $\chi_A : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

χ_A is called the characteristic function of $A \subseteq \mathbb{R}$. Let $P = \{x_0 < x_1 < \ldots < x_n\}$ be a partition of $[a, b]$, and $c_1, c_2, \ldots, c_n \in \mathbb{R}$. A function $f : [a, b] \to \mathbb{R}$ is called a *step function*, if it has the form $f(x) = \sum\limits_{j=1}^{n} c_j \chi_{I_j}$, where $c_j \in \mathbb{R}$, and

I_j are intervals with endpoints x_{j-1} and x_j. Since f is constant on I_j, theorem 7.9 guarantees that f is integrable. Notice that if f is not constant then it is an example of integrable function that is discontinuous.

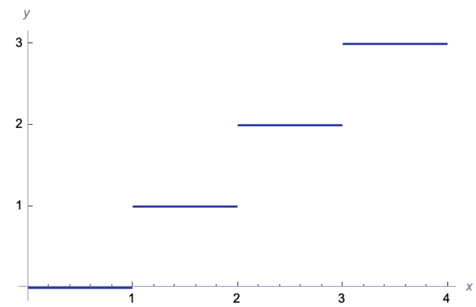


Fig. 7.3: The step function $f(x) = \chi_{(1,2]} + 2\chi_{(2,3]} + 3\chi_{(3,4]}$

Theorem 7.11. *Let $f, g : [a, b] \to \mathbb{R}$ be integrable. Then*

(1) $f + g$ is integrable and $\int_a^b (f + g) = \int_a^b f + \int_a^b g$,
(2) $f \cdot g$ is integrable,
(3) If $\exists k > 0$ such that $0 < k \leq |g(x)|$ for every $x \in [a, b]$, then f/g is integrable,
(4) If $f \leq g$ then $\int_a^b f \leq \int_a^b g$,
(5) $|f|$ is integrable and $\left|\int_a^b f\right| \leq \int_a^b |f|$.

Proof. Notice that for P, Q partitions of $[a, b]$ we have:

$$s(f; P) + s(g; Q) \leq s(f; P \cup Q) + s(g; P \cup Q) \leq s(f + g; P \cup Q) \leq \underline{\int_a^b} (f + g),$$

and hence:

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f + g).$$

Similarly, we can show that $\overline{\int_a^b} f + \overline{\int_a^b} g \geq \overline{\int_a^b} (f + g)$. We conclude from the inequalities

$$\underline{\int_a^b} f + \underline{\int_a^b} g \leq \underline{\int_a^b} (f + g) \leq \overline{\int_a^b} (f + g) \leq \overline{\int_a^b} f + \overline{\int_a^b} g,$$

that (1) is true.

To prove (2), choose $K > 0$ big enough such that $\max\{|f(x)|, |g(x)|\} \leq K$. Let $P = \{x_i; i = 0, \ldots, n\}$ be a partition of $[a, b]$, and $\omega_i', \omega_i'', \omega_i$ the oscillations of f,g and fg respectively, on the interval $[x_i, x_{i-1}]$. For $x, y \in [x_i, x_{i-1}]$ we have:

$$\begin{aligned}|f(y)g(y) - f(x)g(x)| &= |[f(y) - f(x)]g(y) + [g(y) - g(x)]f(x)| \\ &\leq \omega_i' K + \omega_i'' K = (\omega_i' + \omega_i'')K\end{aligned}$$

It follows that:

$$\sum_{k=1}^{n} \omega_i (x_i - x_{i-1}) \leq \sum_{k=1}^{n} (\omega_i' + \omega_i'')K(x_i - x_{i-1}),$$

and (2) is a direct consequence of Theorem 7.7(3).

Item (3) follows from (2), if we can show that $\frac{1}{g}$ is integrable. Let $P = \{x_i; i = 0, \dots, n\}$ be a partition of $[a, b]$, and $x, y \in [x_i, x_{i-1}]$. By hypothesis:

$$\left|\frac{1}{g(x)} - \frac{1}{g(y)}\right| \leq \frac{|g(y) - g(x)|}{k^2}.$$

Once more, the result follows from Theorem 7.7(3).

Item (4) is trivial, since in this case $s(f; P) \leq s(g; P)$ for every partition, hence $\int_a^b f \leq \int_a^b g$. Finally, to see why (5) is true, consider the inequality:

$$||f(x)| - |f(y)|| \leq |f(x) - f(y)|$$

Which tell us that the oscillation of $|f|$ is always bounded by the oscillation of $|f|$, hence by Theorem 7.7(3) again, $|f|$ is integrable. The last part follows from the inequality $-|f(x)| \leq f(x) \leq |f(x)|$. □

Corollary 7.12. *Let $f : [a, b] \to \mathbb{R}$ integrable and bounded, say $|f(x)| \leq K$. Then*

$$\left|\int_a^b f\right| \leq K(b - a).$$

Theorem 7.13. *Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.*

Proof. By Theorem 5.68, f is uniformly continuous. Let $\epsilon > 0$ be given, and take $\delta > 0$ such that $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{b-a}$. Now, choose a partition $P = \{x_i; i = 0, \dots, n\}$ such that $x_i - x_{i-1} < \delta$ for every $i = 1, \dots, n$. If ω_i is the oscillation of f at $[x_{i-1}, x_i]$ then $\omega_i < \frac{\epsilon}{b-a}$ and it follows that

$$\sum_{k=1}^{n} \omega_i (x_i - x_{i-1}) < \frac{\epsilon}{b - a} \sum_{k=1}^{n} (x_i - x_{i-1}) = \epsilon.$$

This completes the proof by Theorem 7.7(3). □

Theorem 7.14. *Let $f : [a, b] \to \mathbb{R}$ be monotone. Then f is integrable.*

Proof. The argument is similar to the above theorem, namely it uses Theorem 7.7(3). Without loss of generality, suppose f increasing. Let $\epsilon > 0$ be given, choose a partition $P = \{x_i; i = 0, \ldots, n\}$ such that $x_i - x_{i-1} < \frac{\epsilon}{f(b)-f(a)}$. We have:

$$\sum_{k=1}^{n} \omega_i (x_i - x_{i-1}) < \frac{\epsilon}{f(b) - f(a)} \sum_{k=1}^{n} \omega_i = \epsilon.$$

□

Example 7.15. Let $\mathbb{R}^+$ be the set of a positive real numbers. The *natural logarithm* function is the function $\ln : \mathbb{R}^+ \longrightarrow \mathbb{R}$ given by

$$\ln x = \int_1^x \frac{1}{x}\, dx.$$

Notice that the function $\frac{1}{x}$ is positive if $x > 0$, thus $\ln x$ is always increasing and hence differentiable and integrable on a closed interval, with $(\ln x)' = \frac{1}{x}$. A quick computation shows that $\ln x$ has all of its derivatives, so it is smooth, i.e. $\ln x \in C^\infty$.

Since $\ln x$ is always increasing, it's injective. We denote its inverse, called the *exponential function*, by $exp(x)$, it's easy to see that $exp(x) = e^x$, where e is the Euler number defined in Example 3.28.

Recall that given an interval $I \subseteq \mathbb{R}$ with end-points a and b, the length of I, denoted by $|I|$, is given by $|I| = b - a$.

A set $X \subseteq \mathbb{R}$ has *measure zero* if given $\epsilon > 0$, it's possible to find a countable open cover of $X \subseteq \bigcup\limits_{n=1}^{\infty} I_n$ by open intervals I_n, such that $\sum\limits_{n=1}^{\infty} |I_n| < \epsilon$.

Example 7.16. Any countable set $X \subseteq \mathbb{R}$ has measure zero. Indeed, given any $\epsilon > 0$, take an open interval of length $\frac{\epsilon}{2^n}$ around the n-th number $x_n \in X$, then $\sum\limits_{n=1}^{\infty} |I_n| < \epsilon$. In particular, the set of Rational numbers $\mathbb{Q}$ has measure zero.

Example 7.17. The Cantor set K has measure zero since after the n-th iteration, K is contained in the union of 2^n intervals of length 3^{-n}. Hence, given any $\epsilon > 0$, if we take n sufficiently large, K can be covered by open sets whose length add to a number less than ϵ.

Theorem 7.18. *(Lebesgue's criterion)Let $f : [a, b] \longrightarrow \mathbb{R}$ be bounded function. The set of discontinuities D of f has measure zero if and only if f is integrable*

Proof. Suppose D has measure zero and $\omega := \sup f - \inf f$ is the oscillation of f in $[a, b]$. Let $\epsilon > 0$ be given, and suppose $D \subseteq \bigcup\limits_{n=1}^{\infty} I_n$, where I_n are open intervals such that $\sum\limits_{n=1}^{\infty} |I_n| < \frac{\epsilon}{2\omega}$. For each $x \in [a, b] - D$, take an interval

$J_x \ni x$, such that the oscillation of f in J_x is less than $\frac{\epsilon}{2(b-a)}$, this is possible because f is continuous at x.

Now, $[a,b] \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cup \left(\bigcup_{x \notin D} J_x\right)$, and by Borel-Lebesgue Theorem, there is a finite subcover, say $I_{n_1} \cup \ldots I_{n_k} \cup J_{x_1} \cup \ldots J_{x_l}$ of $[a,b]$. Form a partition P of $[a,b]$ whose elements are a, b, and each endpoint of I_{n_p} and J_{x_q}, for $p = 1, \ldots k,\ q = 1, \ldots, l$. We write $[y_{j-1}, y_j]$ for an interval associated to P which is contained in I_{n_p}, for some p, and $[y_{t-1}, y_t]$, otherwise. Let ω_j denote the oscillation of f in the j-th interval of P. We have:

$$\begin{aligned} S(f;P) - s(f;P) &= \sum \omega_j (y_j - y_{j-1}) + \sum \omega_t (y_t - y_{t-1}) \\ &< \sum \omega (y_j - y_{j-1}) + \sum \frac{\epsilon}{2(b-a)} (y_t - y_{t-1}) \\ &< \omega \frac{\epsilon}{2\omega} + \frac{\epsilon}{2(b-a)}(b-a) = \epsilon \end{aligned}$$

By Theorem 7.7(3), f is integrable.

Conversely, suppose f is integrable. Set

$$D_n = \left\{ x \in [a,b]\ ;\ \omega(f,x) \geq \frac{1}{n} \right\},$$

thus $D = \bigcup_n D_n$, so it suffices to show that D_n has measure zero. By Theorem 7.7(3), given $n \in \mathbb{N}, \epsilon > 0$ we can find a partition $\{x_i\}$ of $[a,b]$ such that

$$\sum_i \omega_i (x_i - x_{i-1}) < \epsilon \cdot \frac{1}{n}.$$

In the sum above if we consider only the intervals containing points of D_n we obtain $\frac{1}{n}\sum_i (x_i - x_{i-1}) < \sum_i \omega_i (x_i - x_{i-1}) < \epsilon \cdot \frac{1}{n}$, thus $\sum_i (x_i - x_{i-1}) < \epsilon$. The chosen intervals may not cover D_n entirely, since they can miss some points of the partition $\{x_i\}$, but if they do, it would be a finite amount of them and we could add to the already chosen intervals arbitrarily small ones. □

Example 7.19. The Cantor function $f : [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1, & \text{if } x \in K \\ 0, & \text{if } x \notin K, \end{cases}$$

is integrable. Indeed, f is continuous in $[0,1] - K$, but it's discontinuous at every point a of K, since we can find a sequence $x_n \in [0,1] - K$ such that $x_n \to a$. By Theorem 7.18, f is integrable.

Example 7.20. If $a < b$ then $[a,b]$ doesn't have measure zero. Indeed, Let I_n be a open cover of $[a,b]$, by Borel-Lebesgue Theorem, we can extract a finite subcover. After relabeling if necessary, we may assume $[a,b] \subseteq I_1 \cup \ldots \cup I_n$.

Since the I_j are intervals, we have $\bigcup_{j=1}^{n} I_j \subseteq [c, d]$, for some $c, d \in \mathbb{R}$. It follows that $\chi_{[a,b]} \leq \chi_{\cup I_j}$, which implies that

$$b - a = \int_c^d \chi_{[a,b]} \leq \int_c^d \chi_{\cup I_j} = \sum_{j=1}^{n} |I_j|$$

Thus, $\sum_{j=1}^{n} |I_j| > b - a$, in particular it can't be arbitrarily small.

Recall that a point $c \in [a, b]$ is a critical point for the function $f : [a, b] \to \mathbb{R}$ if $f'(c) = 0$. When $y = f(c)$, for some critical point $c \in [a, b]$, we say y is a critical value of f.

Example 7.21. (Riemann's Example) For $x \in \mathbb{R}$, let $\langle x \rangle$ denotes the fractional part of x, i.e. $\langle x \rangle = x - \lfloor x \rfloor$ (see Example 1.24). For each $x \in [0, +\infty)$, consider the series:

$$P(x) := \sum_{n=1}^{\infty} \frac{\langle nx \rangle}{n^2}$$

Since $\langle x \rangle \leq 1$ for every $x \in [0, +\infty)$ and $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, the function $P(x)$ is well defined and $|P(x)| \leq \frac{\pi^2}{6}$. Notice that $P(x)$ is periodic since $P(x+1) = P(x)$. This function is example of a function that is continuous at every irrational number but discontinuous at every rational. It follows from Theorem 7.18 that $P(x)$ is integrable. The graph of $P(x)$ on $[0, 1]$ is shown in Figure 7.4. In Exercise 11, you will show that it's impossible to have a function whose set of discontinuities is the irrationals.

Theorem 7.22. *(Sard) Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Then the set of critical values of f has measure zero.*

Proof. Let X be the set of critical values of f. Fix $\delta > 0$ and define

$$X_\delta := \{x \in [a, b]; |f'(x)| < \delta\}$$

It follows that $X \subseteq f(X_\delta)$.

Since X_δ is open and bounded, by Theorem 4.10, it can be written as a disjoint countable union of open intervals, say $X_\delta = \bigcup_k I_{\delta k}$. Notice that

$$X \subseteq f(X_\delta) = \bigcup_k f(I_{\delta k})$$

Since f is continuous and $I_{\delta k}$ is an interval, $f(I_{\delta k})$ is again an interval, which we may assume open, if not, we remove the endpoints and consider an arbitrarily

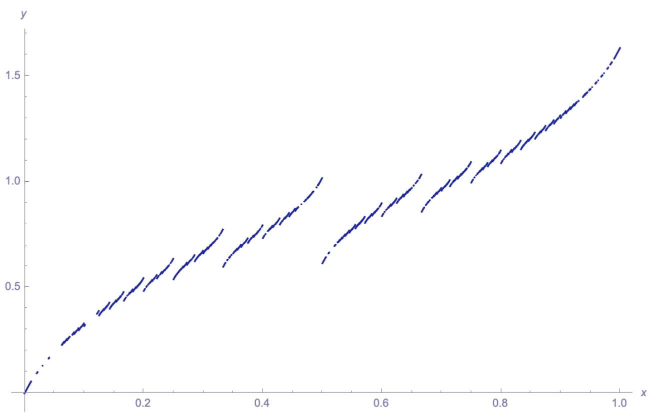


Fig. 7.4: The function $P(x) = \sum_{n=1}^{\infty} \frac{\langle nx \rangle}{n^2}$

small open interval around them. By the mean value theorem, we must have $|f(I_{\delta k})| = |f'(x_{\delta k})||I_{\delta k}| < \delta |I_{\delta k}|$.

Therefore, $\sum_k |f(I_{\delta k})| \leq \delta \sum_k |I_{\delta k}| \leq \delta(b-a)$. Given any $\epsilon > 0$, we may take $\delta = \frac{\epsilon}{b-a}$ and it follows that $\{f(I_{\delta k})\}$ is a cover of X such that $\sum_k f(I_{\delta k}) < \epsilon$. $\square$

7.3 The Fundamental Theorem of Calculus

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. For $x \in [a, b]$ we define:

$$F(x) = \int_a^x f(t)\, dt.$$

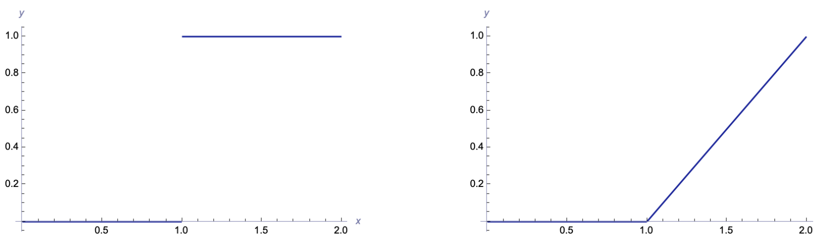
If $f(x)$ is bounded, say $|f(x)| \leq K$ then for $x, y \in [a, b]$:

$$|F(x) - F(y)| \leq \left| \int_y^x f(t)\, dt \right| \leq K|x - y|.$$

Hence, $F(x)$ is Lipschitz, in particular uniformly continuous even if $f(x)$ is only integrable and bounded.

We say that $F(x)$ is the *antiderivative* of $f(x)$.

Example 7.23. Consider the step function $f(x) = \mathcal{X}_{[1,2]}$ defined on the interval $[0, 2]$ and its antiderivative $F(x)$. We can easily see that $f(x)$ is discontinuous but $F(x)$ is continuous.



(a) $f(x) = \chi_{[1,2]}$ (b) $F(x) = \int_0^x f(t)\,dt$

Fig. 7.5: A function with its antiderivative

Theorem 7.24. *Let $f : [a,b] \to \mathbb{R}$ be an integrable function. If $f(x)$ is continuous at $c \in [a,b]$ then $F(x) = \int_a^x f(t)\,dt$ is differentiable at c and $F'(c) = f(c)$.*

Proof. Given $\epsilon > 0$, we can find $\delta > 0$ such that $|t-c| < \delta \Rightarrow |f(t)-f(c)| < \epsilon$. For $0 < h < \delta$:

$$\left|\frac{F(c+h)-F(c)}{h} - f(c)\right| = \frac{1}{h}\left|\int_c^{c+h} [f(t)-f(c)]\,dt\right| \le \frac{1}{h}\int_c^{c+h} |f(t)-f(c)|\,dt \le \frac{1}{h}\epsilon h = \epsilon$$

A similar argument is true when $-\delta < h < 0$, hence $F'(c) = f(c)$. □

Corollary 7.25. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then its indefinite integral $F(x)$ is differentiable and $F'(x) = f(x)$.*

A differentiable function $F(x)$ is called a *primitive* of $f(x)$ if $F'(x) = f(x)$. Corollary 7.25 is the statement that every continuous function defined on a closed interval has a primitive. Moreover, given any two primitives $F(x)$ and $G(x)$ of $f(x)$ we have $(F-G)'(x) = 0$, hence $F(x)$ and $G(x)$ differ by a constant. We conclude:

Corollary 7.26. *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then any primitive of $f(x)$ has the form $F(x) = \int_a^x f(t)\,dt + C$, where $C \in \mathbb{R}$.*

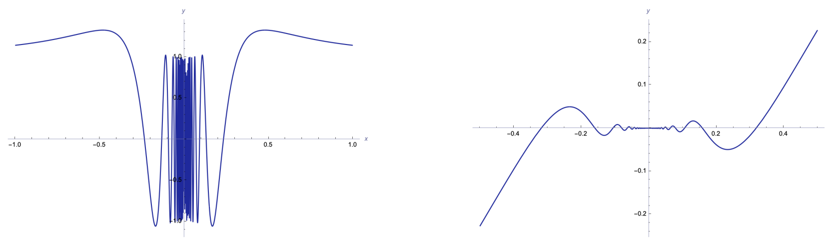
Example 7.27. The function defined by

$$f(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0, \end{cases}$$

is discontinuous at $x = 0$, yet has a primitive given by

$$F(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Therefore, it's possible for a function to have a primitive even if it's discontinuous. Notice that the function in example 7.23 doesn't have a primitive in any interval containing 1.



(a) $f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ (b) $F(x) = x^2 \sin \frac{1}{x}$

Fig. 7.6: A discontinuous function with its primitive

Theorem 7.28. *(Fundamental Theorem of Calculus) Let $f : [a,b] \to \mathbb{R}$ be a differentiable function. If $f'(x)$ is integrable then*

$$\int_a^b f'(t)\, dt = f(b) - f(a)$$

Proof. Let $\{x_i\}$ be a partition of $[a,b]$. By the Mean Value Theorem, there exists $c_i \in [x_{i-1}, x_i]$ such that

$$f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$$

Define $m_i = \inf_{[x_{i-1},x_i]} f'(x)$ and $M_i = \sup_{[x_{i-1},x_i]} f'(x)$. Then $m_i \le f'(c_i) \le M_i$, moreover

$$f(b) - f(a) = \sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = \sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}),$$

it follows that

$$s(f';P) \le f(b) - f(a) \le S(f';P).$$

Since f' is integrable, the numbers $s(f';P)$ and $S(f';P)$ have to be arbitrarily close. The result follows. □

Corollary 7.29. *(Change of Variables) Let $f : [a,b] \to \mathbb{R}$ be a continuous function, $g : [c,d] \to \mathbb{R}$ differentiable with g' integrable, and $g([c,d]) \subseteq [a,b]$. Then*

$$\int_{g(c)}^{g(d)} f(x)\,dx = \int_c^d f(g(t))g'(t)\,dt.$$

Proof. Since f is continuous, it has a primitive, say $F(x)$. Using Theorem 7.28 with the function $F(g(t))$ we obtain

$$\int_c^d f(g(t))g'(t)\,dt = \int_c^d F'(g(t))\,dt = F(g(d)) - F(g(c))$$

On the other hand,

$$\int_{g(c)}^{g(d)} f(x)\,dx = F(g(d)) - F(g(c)).$$

□

Corollary 7.30. *(Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ be functions with integrable derivative,then*

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)]_a^b - \int_a^b f'(x)g(x)dx,$$

where $f(x)g(x)]_a^b = f(b)g(b) - f(a)g(a)$.

Proof. Immediate consequence of the product rule $(fg)' = f'g + fg'$ and theorem 7.28. □

Corollary 7.31. *(Mean Value Theorem - Integral version) Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in (a, b)$ such that*

$$\int_a^b f(x)\,dx = f(c)(b - a)$$

Proof. Let $F(x)$ be a primitive for $f(x)$. Then by the Mean Value Theorem

$$\int_a^b f(x)\,dx = F(b) - F(a) = F'(c)(b - a) = f(c)(b - a).$$

□

Corollary 7.32. *(Taylor's Formula with Integral Remainder) Let $f : [a, a + h] \to \mathbb{R}$ be function having the derivative of order $n + 1$ integrable. Then*

$$f(a+h) = f(a)+f'(a)h+\ldots+\frac{f^{(n)}(a)}{n!}h^n+\left[\int_0^1 \frac{(1-t)^n}{n!} f^{(n+1)}(a+th)\,dt\right] h^{n+1}$$

Proof. Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(a + th)$. It suffices to show that

$$g(1) = g(0) + g'(0) + \ldots + \frac{g^{(n)}(0)}{n!} + \int_0^1 \frac{(1-t)^n}{n!} g^{(n+1)}(t)\, dt$$

If $n = 0$, this is just theorem 7.28. If $n = 1$, using integration by parts we have

$$g(1) = g(0) + \int_0^1 g'(t)\, dt = g(0) + g'(0) + \int_0^1 (1-t)g''(t)\, dt$$

If $n = 2$, using a similar argument we have

$$g(1) = g(0)+g'(0)+\int_0^1 \frac{(1-t)^2}{2} g''(t)\, dt = g(0)+g'(0)+\frac{g''(0)}{2}+\int_0^1 \frac{(1-t)^2}{2} g'''(t)\, dt$$

The proof follows once we Iterate this procedure. □

Gven a partition $P = \{x_i\}$ of $[a, b]$, we define the norm of P, denoted by $|P|$, as

$$|P| := \max_{1\le i\le n} |x_i - x_{i-1}|$$

Theorem 7.33. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Given $\epsilon > 0$, there exists $\delta > 0$, such that*

$$|P| < \delta \Rightarrow S(f; P) < \overline{\int_a^b} f(x)\, dx + \epsilon$$

Proof. It suffices to consider the case where $f(x) \ge 0$, otherwise we could consider $f(x) - \inf f(x) \ge 0$. Let $\epsilon > 0$ be given, then there is a partition $Q = \{x_0, \ldots, x_n\}$, such that

$$S(f; Q) < \overline{\int_a^b} f(x)\, dx + \frac{\epsilon}{2}$$

Let $M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)$ and $M := \sup f(x)$. Take any $\delta > 0$, satisfying $\delta < \frac{\epsilon}{2Mn}$. Let $P = \{y_0, \ldots, y_m\}$ be any partition satisfying $|P| < \delta$, we will use the index 'i' in $[y_{i-1}, y_i]$, whenever $[y_{i-1}, y_i] \subseteq [x_{i-1}, x_i]$, and use the index 'j' for the remaining intervals. We have

$$\begin{aligned} S(f;P) &= \sum_i M_i(y_i - y_{i-1}) + \sum_j M_j(y_j - y_{j-1}) \\ &\le \sum_i M_i(x_i - x_{i-1}) + M\delta n \\ &\le S(f;Q) + \frac{\epsilon}{2} \\ &\le \overline{\int_a^b} f(x)\,dx + \epsilon. \end{aligned}$$

□

The argument above can easily be adapted to prove the equivalent result for $\underline{\int_a^b} f(x)\,dx$.

Corollary 7.34. *Let $f : [a,b] \to \mathbb{R}$ be integrable. Then*

$$\int_a^b f(x)\,dx = \lim_{|P|\to 0} s(f;P) = \lim_{|P|\to 0} S(f;P)$$

A *tagged partition*, denoted by P^*, is a partition $P = \{x_i\}$ together with a collection of points $\{t_i\}$, such that $x_{i-1} \le t_i \le x_i$. Given a function $f : [a,b] \to \mathbb{R}$ and a tagged partition P^* of $[a,b]$, we define the *Riemann sum* of $f(x)$ by

$$R(f;P^*) = \sum_{i=1}^n f(t_i)(x_i - x_{i-1})$$

It follows directly from the definition that

$$s(f;P) \le R(f;P^*) \le S(f;P).$$

Thus, the following corollary is immediate.

Corollary 7.35. *(Integral as a Riemann sum) Let $f : [a,b] \to \mathbb{R}$ be integrable. Then*

$$\int_a^b f(x)\,dx = \lim_{|P^*|\to 0} R(f;P^*)$$

7.4 Improper Integrals

So far we have avoided functions defined on intervals that are not closed. In this section we will discuss the definition of integrals for such functions.

First, we discuss the case where the function is bounded.

Theorem 7.36. *Let $f : (a,b] \to \mathbb{R}$ be bounded. If $f(x)$ is integrable in $[c,b]$ for every $c \in (a,b)$, then can extend $f(x)$ to a function on $[a,b]$ such that*

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

Proof. Take any $v \in \mathbb{R}$ and define $f(a) = v$. Let $K \in \mathbb{R}$ such that $|f(x)| \le M$ for $x \in [a,b]$. By hypothesis, given $\epsilon > 0$, for every $c \in (a,b)$ we can find a partition $\{x_i\}$ of $[c,b]$ such that

$$S(f;P) - s(f;P) < \frac{\epsilon}{2}$$

Choose c such that $M(c-a) < \frac{\epsilon}{4}$. We form a partition of $[a,b]$, say Q, by adding the point a to P. We have

$$S(f;Q) - s(f;Q) \le 2M(c-a) + S(f;P) - s(f;P) < \epsilon,$$

Thus, $f(x)$ is integrable. Moreover, (the negative of) its antiderivative

$$F(x) = \int_x^b f(x)\,dx$$

is Lipschitz as discussed in the beginning of section 7.3, so

$$F(a) = \lim_{c\to a^+} F(c) = \lim_{c\to a^+}\int_c^b f(x)\,dx$$

□

Remark. *377*

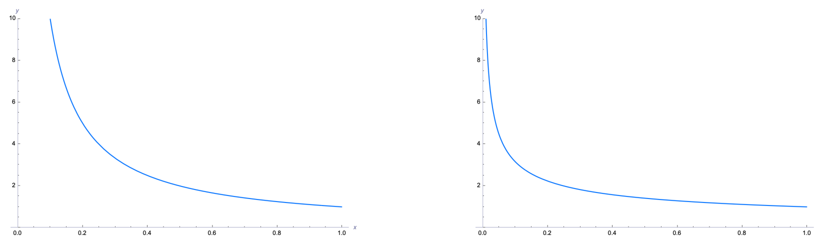
The exact same result is valid if we consider an interval of the form $[a,b)$ instead. Motivated by theorem 7.36, if $f : (a,b] \to \mathbb{R}$ is continuous but unbounded, we define

$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

It's possible that the limit above doesn't exist, in that case we say the integral *diverges* or it's *divergent*. Otherwise, we say the integral *converges* or it's *convergent*. A equivalent definition can be given when is defined on $[a,b)$, namely $\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx$. Lastly, if $f : (a,b) \to \mathbb{R}$ is continuous, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 7.37. Fix $a \in \mathbb{R}$ and consider the function $f : (0, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^a}$.



(a) $f(x) = \frac{1}{x}$ (b) $f(x) = \frac{1}{\sqrt{x}}$

Fig. 7.7: A function with singularity at zero.

Suppose $a \neq 1$, then by definition:

$$\int_0^1 \frac{1}{x^a}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x^a}\,dx = \lim_{c\to 0^+} \frac{x^{1-a}}{1-a}\Big]_c^1 = \begin{cases} \frac{1}{1-a}, & \text{if } a < 1 \\ +\infty, & \text{if } a > 1. \end{cases}$$

When $a = 1$, we obtain

$$\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0^+} \ln x\Big]_c^1 = +\infty.$$

Example 7.38. In some case we don't even have to use the limit definition, just algebraic manipulations and/or integration by parts suffice. For example, consider the unbounded function $f : (0, \frac{\pi}{2}] \longrightarrow \mathbb{R}$ defined by $f(x) = \ln(\sin x)$.

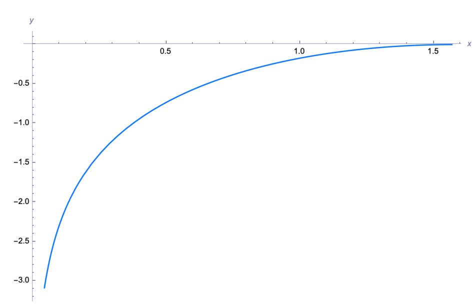


Fig. 7.8: $f(x) = \ln(\sin x)$

First notice that

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\cos x)\,dx,$$

hence

$$\begin{aligned} 2\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx &= \int_0^{\frac{\pi}{2}} \ln(\sin x\cos x)\,dx = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\right)\,dx + \int_0^{\frac{\pi}{2}} \ln(2\sin x\cos x)\,dx \\ &= -\frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx \end{aligned}$$

On the other hand,

$$\int_0^{\frac{\pi}{2}} \ln(\sin 2x)\,dx = \frac{1}{2}\int_0^{\pi} \ln(\sin x)\,dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx + \frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} \ln(\sin x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx$$

Therefore,

$$\int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx = -\frac{\pi}{2}\ln 2,$$

and the integral is convergent.

The following proposition is immediate from the definitions.

Proposition 7.39. *(Comparison Principle) Let $f, g : (a,b] \to \mathbb{R}$ be nonnegative functions. If there exists $k > 0$ such that*

$$0 \le f(x) \le kg(x),$$

and moreover, $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ also converges.

Example 7.40. We claim that the integral

$$\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$$

converges. Indeed, notice that for $0 \le x \le 1$ we have $\frac{x^2}{\sqrt{1-x^2}} \le \frac{1}{\sqrt{1-x^2}}$, so it suffices to prove that $\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$ converges, which is straightforward:

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{c\to 1^-}\int_0^c \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{c\to 1^-} \arcsin c = \frac{\pi}{2}$$

Given a function $f : (a,b] \to \mathbb{R}$, we say $\int_a^b f(x)\,dx$ is *absolutely convergent* if $\int_a^b |f(x)|\,dx$ converges. Similar to the case of series, absolute convergence implies convergence and we have

Proposition 7.41. *(Comparison Principle– absolute convergence) Let $f, g : (a, b] \to \mathbb{R}$ be given. If there exists $k > 0$ such that*

$$|f(x)| \leq kg(x),$$

and moreover, $\int_a^b g(x)\,dx$ converges, then $\int_a^b |f(x)|\,dx$ converges, in particular $\int_a^b f(x)\,dx$ also converges.

Now, we extend the definition of integral to functions defined on unbounded intervals. Let $f : [a, +\infty) \to \mathbb{R}$ be continuous. We define

$$\int_a^{+\infty} f(x)\,dx = \lim_{n\to+\infty} \int_a^n f(x)\,dx,$$

as before, if the limit exists we say the integral converges, otherwise, we say it diverges. Similar definitions can be given when $f(x)$ is defined on $(-\infty, b]$ or $(-\infty, +\infty)$.

Example 7.42. Let's revisit example 7.37. Suppose $f : [1, +\infty) \to \mathbb{R}$ is given by $f(x) = \frac{1}{x^a}$ for a fixed $a \in \mathbb{R}$. If $a \neq 1$ we have

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x^a}\,dx &= \lim_{n\to+\infty} \int_1^n \frac{1}{x^a}\,dx \\ &= \lim_{n\to+\infty} \frac{x^{1-a}}{1-a}\Big]_1^n \\ &= \lim_{n\to+\infty} \frac{n^{1-a}-1}{1-a} = \begin{cases} \frac{1}{a-1}, & \text{if } a > 1 \\ +\infty, & \text{if } a < 1. \end{cases}\end{aligned}$$

When $a = 1$, we have

$$\int_1^{+\infty} \frac{1}{x}\,dx = \lim_{n\to+\infty} \int_1^n \frac{1}{x}\,dx = \lim_{n\to+\infty} \ln n = +\infty$$

As before, the comparison principle is also valid in this case. For the sake of completeness we write below.

Proposition 7.43. *(Comparison Principle) Let $f, g : [a, +\infty) \to \mathbb{R}$ be given. If there exists $k > 0$ such that*

$$|f(x)| \leq kg(x),$$

and moreover, $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} |f(x)|\,dx$ converges, in particular $\int_a^{+\infty} f(x)\,dx$ also converges.

Example 7.44. Despite the periodic behavior of $\sin x^2$, the integral

$$\int_0^\infty \sin x^2 \, dx$$

is actually convergent.

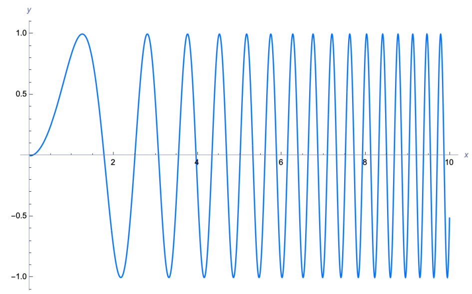


Fig. 7.9: $f(x) = \sin x^2$

Indeed, since $\sin x^2$ is integrable on $[0, 1]$, it suffices to analyze the integral on $[1, +\infty)$. We have

$$\int_1^\infty \sin x^2 \, dx = \lim_{n\to+\infty} \int_1^n \sin x^2 \, dx = \lim_{n\to+\infty} \int_1^{n^2} \frac{\sin x}{2\sqrt{x}} \, dx.$$

Integrating by parts the last integral we have

$$\int_1^{n^2} \frac{\sin x}{2\sqrt{x}} \, dx = -\frac{\cos x}{2\sqrt{x}}\Big]_1^{n^2} - \int_1^{n^2} \frac{\cos x}{4x^{\frac{3}{2}}} \, dx,$$

taking the limit we obtain

$$\lim_{n\to+\infty} \int_1^{n^2} \frac{\sin x}{2\sqrt{x}} \, dx = \frac{\cos 1}{2} - \frac{1}{4} \int_1^\infty \frac{\cos x}{x^{\frac{3}{2}}} \, dx,$$

but $\left|(\cos x)x^{-\frac{3}{2}}\right| \leq x^{-\frac{3}{2}}$, and by example 7.42 we know that $\int_1^\infty \frac{1}{x^{\frac{3}{2}}} \, dx$ converges. Hence, $\int_0^\infty \sin x^2 \, dx$ converges. The actual value of the integral is $\sqrt{\frac{\pi}{8}}$.

Example 7.45. Consider the integral

$$\int_0^{+\infty} \frac{1}{(1+x)\sqrt{x}} = \int_0^1 \frac{1}{(1+x)\sqrt{x}} + \int_1^{+\infty} \frac{1}{(1+x)\sqrt{x}}$$

We have

$$\int_0^1 \frac{1}{(1+x)\sqrt{x}} = \lim_{c\to 0^+} \int_c^1 \frac{1}{(1+x)\sqrt{x}}$$
$$= \lim_{c\to 0^+} 2\arctan\sqrt{x}\Big]_c^1$$
$$= \frac{\pi}{2}$$

Similarly,

$$\int_1^{+\infty} \frac{1}{(1+x)\sqrt{x}} = \lim_{n\to +\infty} \int_1^n \frac{1}{(1+x)\sqrt{x}}$$
$$= \lim_{n\to +\infty} 2\arctan\sqrt{x}\Big]_1^n$$
$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

Thus,

$$\int_0^{+\infty} \frac{1}{(1+x)\sqrt{x}} = \pi.$$

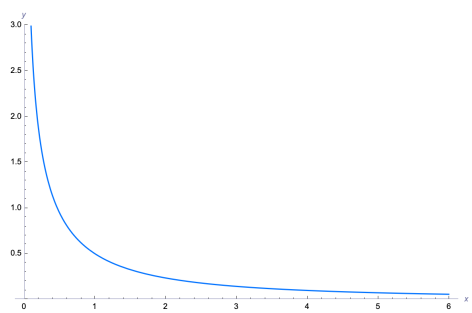


Fig. 7.10: $f(x) = \frac{1}{(1+x)\sqrt{x}}$

Theorem 7.46. *(Integral test) Let $a \in \mathbb{Z}$ and $f : [a, +\infty) \to \mathbb{R}$ a decreasing function. Define for every natural $n \geq a$*

$$a_n = f(n).$$

The series $\sum a_n$ converges if and only if $\int_a^\infty f(x)\,dx$ converges.

Proof. Since f is decreasing, it follows from theorem 7.14 that f integrable on every closed interval. For $x \in [n, +\infty)$ we have

$$f(x) \leq f(n).$$

Similarly, in $(-\infty, n]$ we have

$$f(n) \le f(x).$$

Hence, for every $n \ge a$, we obtain

$$\int_n^{n+1} f(x)\,dx \le \int_n^{n+1} f(n)\,dx = f(n),$$

and for $n \ge a+1$

$$f(n) = \int_{n-1}^{n} f(n)\,dx \le \int_{n-1}^{n} f(x)\,dx.$$

We conclude that

$$\int_n^{n+1} f(x)\,dx \le f(n) \le \int_{n-1}^{n} f(x)\,dx.$$

By summing over all n from a to a fixed integer $m > n$, we obtain

$$\int_a^{m+1} f(x)\,dx \le \sum_{n=a}^{m} f(n) \le f(a) + \int_a^{m} f(x)\,dx.$$

The conclusion follows by letting $m \to +\infty$. □

Example 7.47. Fix $p \in \mathbb{R}$ and consider the series $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^p}$. The corresponding integral is

$$\int_2^{\infty} \frac{1}{x(\ln x)^p}\,dx$$

We can easily compute the integral above using substitution (change of variables):

$$\int_2^{+\infty} \frac{1}{x(\ln x)^p}\,dx = \frac{1}{1-p}(\ln x)^{1-p}\Big]_2^{+\infty}, \text{ for } p \ne 1$$

$$\int_2^{+\infty} \frac{1}{x\ln x}\,dx = \ln(\ln x)\Big]_2^{+\infty}, \text{ for } p = 1$$

Using the integral test, it follows that $\sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^p}$ converges if $p > 1$, and diverges if $p \le 1$.

Exercises

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Prove that if f is not identically zero then $\int_a^b |f(x)|\, dx > 0$.
2. Give an example of an integrable function that is discontinuous at an infinite set.
3. (Cauchy-Schwarz inequality) Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Show that

$$\left|\int_a^b f(x)g(x)\, dx\right|^2 \le \left(\int_a^b f(x)^2\, dx\right)\left(\int_a^b g(x)^2\, dx\right)$$

4. Let $f : [a, b] \to \mathbb{R}$ be a nonnegative integrable function. Consider the set

$$X := \{\phi : [a, b] \to \mathbb{R}\,;\ \phi \text{ is a step function and } \phi(x) \le f(x)\ \forall x\}$$

Show that $\underline{\int_a^b} f(x)\, dx = \sup\limits_{\phi \in X} \int_a^b \phi(x)\, dx$. Show that the result is still valid if we replace the condition 'step function' by continuous or integrable function.

5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable satisfying

$$f(0) = 0 \text{ and } f'(x) = |f(x)|^2$$

Show that $f(x)$ is identically zero.

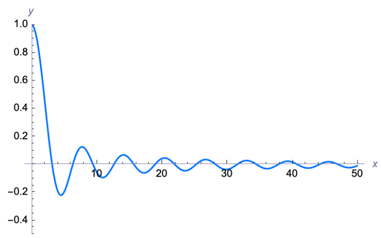
6. Let $f(x) = \int_1^x \frac{\ln t}{1+t}\, dt$. Find the value of $c \in \mathbb{R}$ such that

$$f(x) + f\left(\frac{1}{x}\right) = c(\ln x)^2.$$

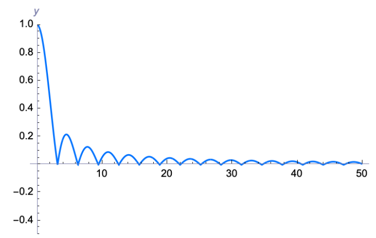
7. Give an example of a non integrable function that has a primitive.
8. Suppose $f : [0, 2] \to \mathbb{R}$ and $g : [-1, 1] \to \mathbb{R}$ are integrable. Show that

$$\int_0^2 (x-1) f(x-1)^2\, dx = 0 = \int_0^\pi g(\sin x)\cos x\, dx$$

9. Show that $\int_0^\infty \frac{\sin x}{x}\, dx$ converges but $\int_0^\infty \left|\frac{\sin x}{x}\right| dx$ doesn't.
10. Let $\alpha \notin \mathbb{N}$ and consider the function $f(x) = (1 + x)^\alpha$. Show that the Taylor series of $f(x)$ around zero converges if $x \in (-1, 1)$.
11. Show that there can't be a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous only at the rational numbers. *Hint: Use Baire Category Theorem (Chapter 4,Exe. 34)*
12. Show that if an interval has measure zero then it's either empty of consists of a single point.
13. Show that every set with empty interior has measure zero.



(a) $f(x) = \frac{\sin x}{x}$



(b) $f(x) = \left|\frac{\sin x}{x}\right|$

Fig. 7.11: Problem 7

14. Let $f : [a, b] \longrightarrow \mathbb{R}$ be a Lipschitz function. Show that if $X \subseteq [a, b]$ has measure zero then $f(X)$ also has measure zero.
15. Let K be the Cantor set. Give an example of a continuous monotone function $f : [0, 1] \to [0, 1]$ such that $f(K)$ doesn't have measure zero.
16. Let $g : [a, b] \longrightarrow \mathbb{R}$ be a nonnegative integrable function such that $\int_a^b g(x)\, dx = 0$. Show that for every integrable $f(x)$, we have $\int_a^b f(x)g(x)\, dx = 0$.
17. If X has measure zero does it follow that $\overline{X}$ also has measure zero?
18. Find two disjoints sets such that $\mathbb{R} = X \cup Y$, X has measure zero and Y is a *meager set* (countable union of closed sets with empty interior).
19. Show that

$$\int_0^\infty \frac{\cos x}{1+x}\, dx = \int_0^\infty \frac{\sin x}{(1+x)^2}\, dx$$

20. Let $1 < s < \infty$. We define the *Riemann's Zeta function* by

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}$$

Show that

$$\zeta(x) = x \int_1^\infty \frac{\lfloor t \rfloor}{t^{x+1}}\, dt.$$

(The floor function $\lfloor x \rfloor$ is defined in Example 1.24)

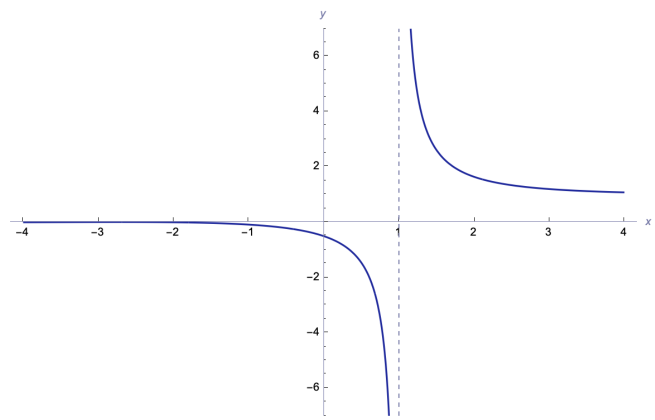


Fig. 7.12: The Zeta function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$

Chapter 8
Sequences and series of functions

8.1 Pointwise and uniform convergence

Let $X \subseteq \mathbb{R}$ be subset of real numbers. A sequence of function $f_n : X \longrightarrow \mathbb{R}$ *converges pointwise* to a function $f : X \longrightarrow \mathbb{R}$, denoted by $f_n \longrightarrow f$, if for every $x \in X$, the sequence of real numbers $f_n(x)$ converges to $f(x)$, i.e.

$$\lim_{n\to+\infty} f_n(x) = f(x).$$

Notice that the limit is with respect to n, and x is fixed, hence the term *pointwise*.

Example 8.1. The sequence $f_n : (0,1) \longrightarrow \mathbb{R}$, given by $f_n(x) = |\sin x|^{-\frac{n}{x}}$ converges pointwise to the constant function $f : (0,1) \longrightarrow \mathbb{R}$ defined by $f \equiv 0$.

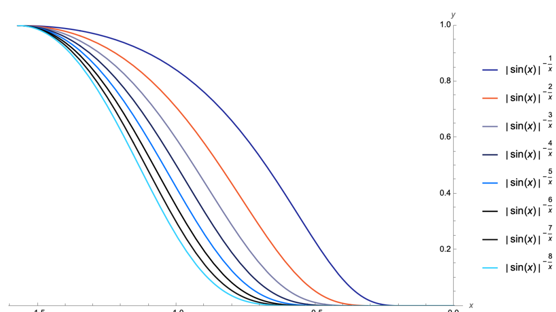


Fig. 8.1: $f_n(x) = |\sin x|^{-\frac{n}{x}}$ for $1 \le n \le 8$.

Indeed, if we fix $x \in (0,1)$, then $\lim_{n\to+\infty} |\sin x|^{-\frac{n}{x}} = 0$.

Example 8.2. Consider the sequence $f_n : \mathbb{R} \longrightarrow \mathbb{R}$, given by $f_n(x) = \frac{|x|}{n}$.

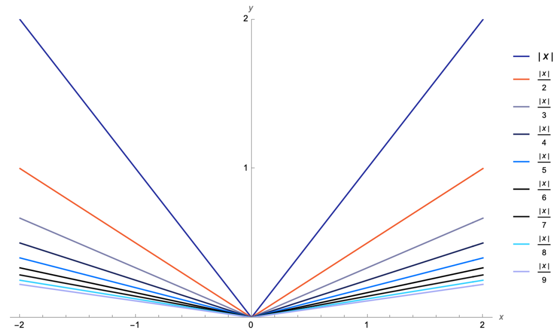


Fig. 8.2: $f_n(x) = \frac{|x|}{n}$ for $1 \leq n \leq 8$.

For a fixed $x \in \mathbb{R}$, we clearly have $\lim\limits_{n\to+\infty} \frac{|x|}{n} = 0$, thus f_n converges pointwise to the constant function $f(x) = 0$.

Example 8.3. Let $f_n : [0, \pi] \to \mathbb{R}$, given by $f_n(x) = \frac{\sin nx}{n}$.

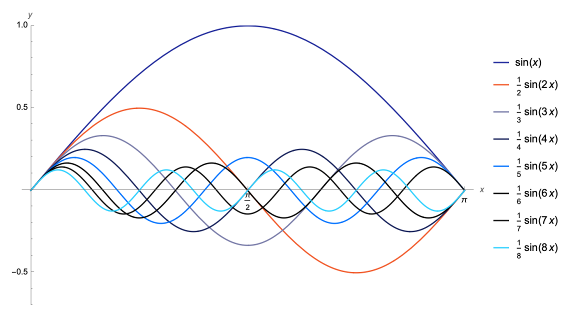


Fig. 8.3: $f_n(x) = \frac{\sin nx}{n}$ for $1 \leq n \leq 8$.

For any $x \in [0, \pi]$,

$$-\frac{1}{n} \leq \frac{\sin nx}{n} \leq \frac{1}{n}.$$

By the Squeeze Theorem we have

$$\lim_{n\to+\infty} \frac{\sin nx}{n} = 0,$$

and we conclude that f_n converges pointwise to 0.

Example 8.4. Suppose $f_n : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by $f_n(x) = \ln\left(\frac{|x|}{n}\right)$.

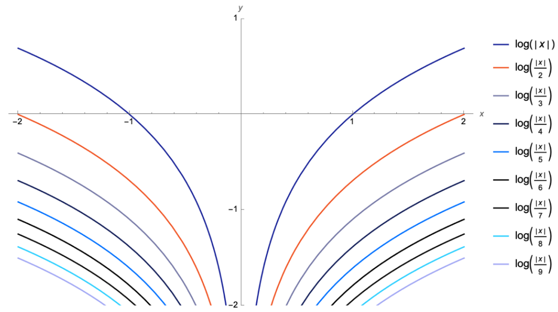


Fig. 8.4: $f_n(x) = \ln\left(\frac{|x|}{n}\right)$ for $1 \leq n \leq 9$.

We claim f_n doesn't converge pointwise, indeed, we have for any $x \neq 0$,

$$\lim_{n\to+\infty} \ln\left(\frac{|x|}{n}\right) = -\infty.$$

We now introduce a stronger notion of convergence, that was not discussed in chapter 3.

A sequence of functions $f_n : X \longrightarrow \mathbb{R}$ *converges uniformly* to a function $f : X \longrightarrow \mathbb{R}$, if given $\epsilon > 0$ there exists $n_0 > 0$ such that

$$n > n_0 \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

In other words, the graph of f_n is arbitrarily close to the graph of f in the sense that for $n > n_0$, we have $f_n(x) \in (f(x) - \epsilon, f(x) + \epsilon)$. Notice that n_0 doesn't depend on x. In particular we have:

Proposition 8.5. *If $f_n \longrightarrow f$ uniformly then $f_n \longrightarrow f$ pointwise.*

It follows directly from the definition that the convergence in Example 8.2 is not uniform but the one in Example 8.3 is. Here's another Example:

Example 8.6. The sequence $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f_n(x) = \frac{x}{1+n^2x^2}$ converges uniformly to zero ($f \equiv 0$).

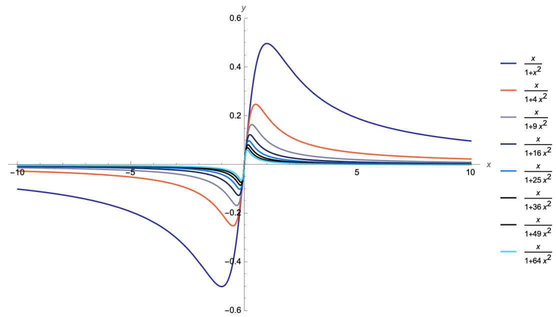


Fig. 8.5: $f_n(x) = \frac{x}{1+n^2x^2}$ for $1 \leq n \leq 9$.

Indeed, first notice that f_n has a global maximum(minimum) at $x = \frac{1}{n}(-\frac{1}{n})$, with corresponding values $\frac{1}{2n}, -\frac{1}{2n}$. Let $\epsilon > 0$ be given. Choose $n > \frac{1}{2\epsilon}$ then

$$\frac{|x|}{1+n^2|x|^2} \leq \frac{1}{2n} < \epsilon$$

as required.

Example 8.7. The sequence $f_n : [0,1] \to \mathbb{R}$ given by $f_n(x) = x^n(1-x^n)$ does not converge uniformly to zero because f_n has a global maximum at $x = \sqrt[n]{\frac{1}{2}}$, with $f\left(\sqrt[n]{\frac{1}{2}}\right) = \frac{1}{4}$. Therefore, given any $\epsilon < \frac{1}{4}$, there is no number $n \in \mathbb{N}$ such that $f_n < \epsilon$. Notice that by the squeeze Theorem f_n does converge pointwise to zero.

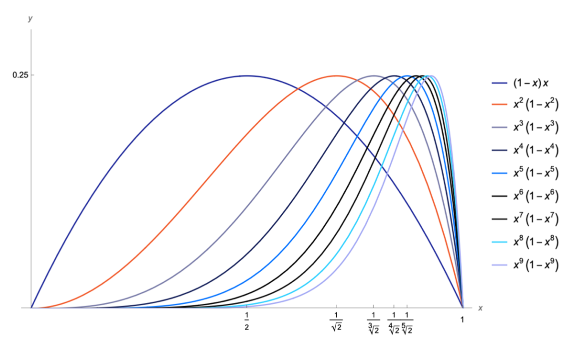


Fig. 8.6: $f_n(x) = x^n(1-x^n)$ for $1 \leq n \leq 8$.

Example 8.8. Let $f_n : (0,1] \to \mathbb{R}$ be given by $f_n(x) = \sin(\ln x^n)$, we can easily see that f_n doesn't even converge pointwise, hence it can't converge uniformly to any function $f_n : (0,1] \to \mathbb{R}$.

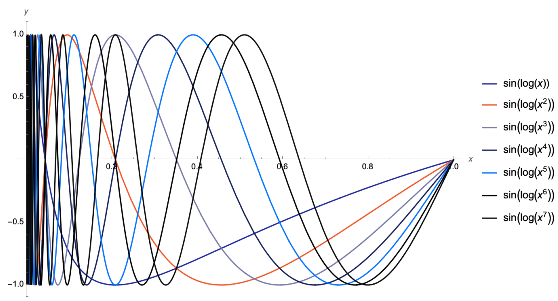


Fig. 8.7: $f_n(x) = \sin(\ln x^n)$ for $1 \leq n \leq 7$.

8.2 Series of functions

Let $f_n : X \to \mathbb{R}$ be a sequence of functions. For each $x \in X$, we may consider the series of real numbers

$$\sum_{n=1}^{\infty} f_n(x),$$

If this series converges for every $x \in X$, it defines a function $f(x) : X \to \mathbb{R}$ given by

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

and we say the series $\sum_{n=1}^{\infty} f_n$ converges to the function f.

Equivalently, define the partial sums

$$s_n(x) = \sum_{j=1}^{j=n} f_j(x).$$

Then the series $\sum_{n=1}^{\infty} f_n$ converges to f if

$$\lim_{n\to\infty} s_n(x) = f(x) \text{ for every } x \in X.$$

We say $\sum_{n=1}^{\infty} f_n$ *converges uniformly* to f if if the sequence of partial sums s_n converges uniformly to f.

Given that the definition of a series of functions relies on sequences, it is natural to expect that a result for sequences has a corresponding counterpart for series.

A sequence of functions $f_n : X \to \mathbb{R}$ is a *Cauchy sequence* if given $\epsilon > 0$, there exists $n_0 > 0$ such that

$$n, m > n_0 \Rightarrow |f_n(x) - f_m(x)| < \epsilon \text{ for every } x \in X$$

Theorem 8.9. *(Cauchy's criterion) A sequence of functions $f_n : X \to \mathbb{R}$ is uniformly convergent if and only if it is a Cauchy sequence.*

Proof. Suppose f_n converges to f uniformly. Given $\epsilon > 0$, we can find n_0 such that

$$n > n_0 \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

In particular, if $m, n > n_0$ then

$$|f_n(x) - f_m(x)| \leq |f_n(x) - f(x)| + |f(x) - f_m(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

thus f_n is Cauchy.

Conversely, suppose f_n Cauchy. Then, by Theorem 3.43, the sequence of numbers $f_n(x)$ is convergent. Define $f(x) = \lim_{n\to+\infty} f_n(x)$, we claim that this convergence is uniform. Given $\epsilon > 0$, there exists n_0 such that $n, m > n_0 \Rightarrow |f_n(x) - f_m(x)| < \epsilon$. Fix n and let $m \to +\infty$ we obtain

$$n > n_0 \Rightarrow |f_n(x) - f(x)| < \epsilon,$$

Hence, $f_n \to f$ uniformly as desired. □

Corollary 8.10. *(Weierstrass M-test*) Suppose $f_n : X \to \mathbb{R}$ is a sequence of functions satisfying*

$$|f_n(x)| \leq a_n,$$

where a_n is a sequence of non-negative real numbers ($a_n \geq 0$). If $\sum_{n=1}^{\infty} a_n$ converges, then both

$$\sum_{n=1}^{\infty} |f_n| \text{ and } \sum_{n=1}^{\infty} f_n$$

converge uniformly.

Proof. For $m, n \in \mathbb{N}$ and arbitrary $x \in X$, we have

* The 'M' in the M-test stands for majorant. The test was originally called the Weierstraßsche Majorantenkriterium, named after the German mathematician Karl Weierstrass.

$$|f_n(x) + \ldots f_{n+m}(x)| \leq |f_n(x)| + \ldots + |f_{n+m}(x)| \leq a_n + \ldots + a_{n+m} \qquad (8.1)$$

Since $\sum_{n=1}^{\infty} a_n$ converges, its partial sums are Cauchy. Hence, by (8.1), the partial sums of $\sum_{n=1}^{\infty} f_n$ and $\sum_{n=1}^{\infty} |f_n|$ are Cauchy, by Theorem 8.9, they both converge uniformly. □

Example 8.11. Let's analyze the convergence of $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

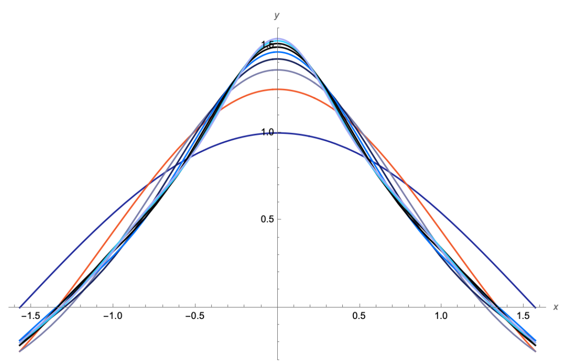


Fig. 8.8: Partial sums $s_k = \sum_{n=1}^{k} \frac{\cos nx}{n^2}$ for the series $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

First, notice that

$$\left|\frac{\cos nx}{n^2}\right| \leq \frac{1}{n^2}.$$

We already know that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (Example 3.63). Hence, by the Weierstrass M-test, $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent.

Example 8.12. The series $\sum_{n=2}^{+\infty} \frac{\arctan x^n}{n(n-1)}$ is convergent.

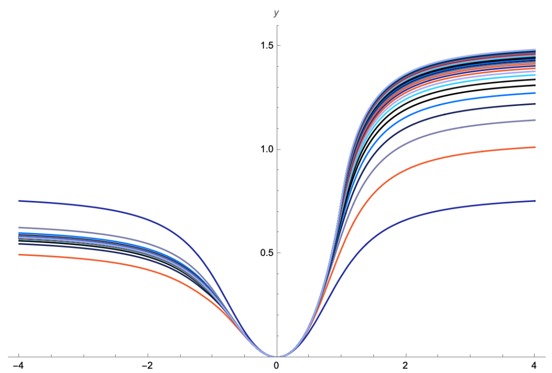


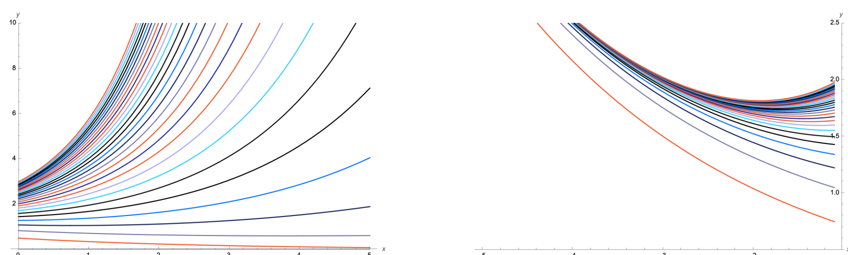
Fig. 8.9: Partial sums s_k (k=2 to 30) for the series $\sum_{n=2}^{+\infty} \frac{\arctan x^n}{n(n-1)}$

Indeed, we observe that

$$\left|\frac{\arctan x^n}{n(n-1)}\right| \leq \frac{\pi}{2n(n-1)}.$$

The series $\sum_{n=2}^{+\infty} \frac{\pi}{2n(n-1)}$ is convergent (Example 3.58). Therefore, by the Weierstrass M-test, $\sum_{n=2}^{+\infty} \frac{\arctan x^n}{n(n-1)}$ is uniformly convergent.

Example 8.13. Consider the series $\sum_{n=2}^{+\infty} \frac{(\ln n)^x}{n}$ for $x \in \mathbb{R}$.



(a) $x \geq 0$ (b) $x < -1$

Fig. 8.10: Partial sums s_k (k=1 to 30) for the series $\sum_{n=2}^{+\infty} \frac{(\ln n)^x}{n}$

We are tempted to use the M-Test, but in fact, for $x \geq 0$, $\frac{(\ln n)^x}{n}$ is bounded from below

$$\left|\frac{(\ln n)^x}{n}\right| \geq \frac{(\ln 2)^x}{n}$$

Using the fact that the harmonic series $\sum_{n=2}^{+\infty} \frac{1}{n}$ diverges, we obtain that $\sum_{n=2}^{+\infty} \frac{(\ln n)^x}{n}$ is divergent for $x \geq 0$. In fact, this Example is the same as Example 7.47, and the series diverges for $x \geq -1$, and converges for $x < -1$.

Theorem 8.14. *Let $f_n : X \to \mathbb{R}$ be a sequence of function converging uniformly to $f : X \to \mathbb{R}$. For $a \in X'$, if $\lim_{x\to a} f_n(x)$ exists for every $n \in \mathbb{N}$, then $\lim_{n\to+\infty}\left[\lim_{x\to a} f_n(x)\right]$ exists. Moreover,*

$$\lim_{n\to+\infty}\left[\lim_{x\to a} f_n(x)\right] = \lim_{x\to a}\left[\lim_{n\to+\infty} f_n(x)\right].$$

Proof. Let $x_n = \lim_{x\to a} f_n(x)$. Suppose $\epsilon > 0$ is given, then there exists $n_0 \in \mathbb{N}$ such that

$$n, m > n_0 \Rightarrow |f_n(x) - f_m(x)| < \frac{\epsilon}{3}$$

For any $n, m > n_0$, it's possible to find $c \in X$ such that $|x_m - f_m(c)| < \frac{\epsilon}{3}$ and $|x_n - f_n(c)| < \frac{\epsilon}{3}$. It follows that

$$|x_n - x_m| \leq |x_n - f_n(c)| + |f_m(c) - x_m| + |f_n(c) - f_m(c)| < \epsilon.$$

Thus, x_n is a Cauchy sequence, hence convergent, say $\lim_{n\to+\infty} x_n = L$. It remains to be proved that $L = \lim_{x\to a} f(x)$. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow |L - x_n| < \frac{\epsilon}{3} \text{ and } |f_n(x) - f(x)| < \frac{\epsilon}{3}.$$

Since $x_n = \lim_{x\to a} f_n(x)$, there exists $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f_n(x) - x_n| < \frac{\epsilon}{3}.$$

Fix $n > n_0$, then for $0 < |x - a| < \delta$, we obtain

$$|f(x) - L| < |f(x) - f_n(x)| + |f_n(x) - x_n| + |x_n - L| < \epsilon.$$

□

The following corollaries are immediate consequences of the Theorem above.

Corollary 8.15. *If $f_n \to f$ uniformly in X and f_n are continuous at $a \in X$, then f is continuous at a. Hence, if f_n are continuous for every $n \in \mathbb{N}$ then f is continuous as well.*

Corollary 8.16. *If the series $\sum_{n=1}^{\infty} f_n$ converge uniformly to f in X, and $\lim_{x\to a} f_n(x)$ exists for every $n \in \mathbb{N}$, then $\sum_{n=1}^{\infty} \lim_{x\to a} f_n(x)$ converges and we have*

$$\lim_{x\to a}\left[\sum_{n=1}^{\infty} f_n(x)\right] = \sum_{n=1}^{\infty}[\lim_{x\to a} f_n(x)]$$

Example 8.17. Consider the sequence $f_n(x) = x^n$ on $[0, 1]$. It converges pointwise to the discontinuous function

$$f(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{if } x \neq 1 \end{cases}$$

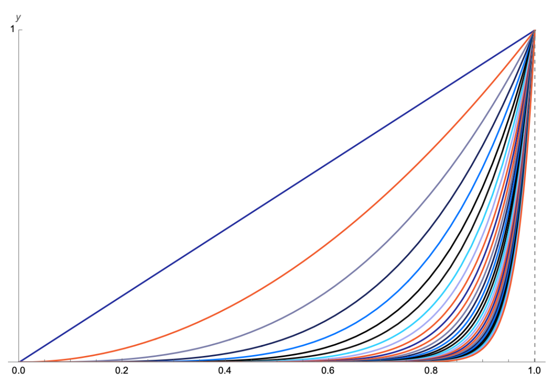


Fig. 8.11: The sequence $f_n(x) = x^n$

Since f is discontinuous, the convergence can't be uniform by Corollary 8.15.

If f is continuous and $f_n \to f$ pointwise in X, the convergence is not necessarily uniform, that is, the continuity of the limit function f is necessary but not sufficient for uniform convergence of a sequence of continuous functions. However, the compactness of X and monotonicity of f_n are sufficient for uniform convergence, as the following Theorem shows.

Theorem 8.18. *(Dini) Let $X \subseteq \mathbb{R}$ be compact, and $f_n : X \to \mathbb{R}$ a sequence of continuous functions such that, for each $x \in X$, the sequence $f_1(x), f_2(x), \ldots$ is monotone. If $f_n \to f$ pointwise and f is continuous, the convergence $f_n \to f$ is uniform.*

Proof. Given $\epsilon > 0$, for each $n \in \mathbb{N}$, define $X_n = \{x \in X;\ |f_n(x) - f(x)| \geq \epsilon\}$. Since both f_n and f are compact, the set X_n is closed, hence compact. Moreover, since f_n is monotone we have $|f_{n+1}(x) - f(x)| \leq |f_n(x) - f(x)|$, thus $X_{n+1} \subseteq X_n$. Let $E = \bigcap_{n=1}^{+\infty} X_n$, if $x \in E$ then $|f_n(x) - f(x)| \geq \epsilon$ for every $n \in \mathbb{N}$, this is impossible since $f_n \to f$ pointwise. It follows that $E = \emptyset$, hence $X_{n_0} = \emptyset$ for some $n_0 \in \mathbb{N}$, and

$$n > n_0 \Rightarrow |f_n(x) - f(x)| < \epsilon.$$

□

Corollary 8.19. *Let $f_n : X \to \mathbb{R}$ be a sequence of nonnegative (or nonpositive) functions, where X is a compact set. Then $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f : X \to \mathbb{R}$ if and only if f is continuous.*

Proof. Indeed, the partial sums $s_k = \sum_{n=1}^{k} f_n$ form a monotone sequence. □

Notice that the sequence f_n from Example 8.17 is monotone and converges pointwise to the zero function on the non-compact interval $[0, 1)$, but the convergence is not uniform (since $\lim_{x \to 1^-} x^n = 1$). Theorem 8.18 cannot be applied in this case.

The next result tells us when it is valid to interchange the limit and the integral for a sequence of functions.

Theorem 8.20. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of integrable functions. If $f_n \to f$ uniformly then $f : [a, b] \to \mathbb{R}$ is integrable and*

$$\int_a^b f(x)\,dx = \int_a^b \lim_{n \to +\infty} f_n(x)\,dx = \lim_{n \to +\infty} \int_a^b f_n(x)\,dx$$

Corollary 8.21. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions. If $\sum_{n=1}^{\infty} f_n$ converges uniformly to a function $f : [a, b] \to \mathbb{R}$ then f is integrable and*

$$\int_a^b f(x)\,dx = \int_a^b \sum_{n=1}^{\infty} f_n(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx$$

Example 8.22. Let $|x| < 1$. Consider the series

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots + x^n + \ldots$$

Using the M-Test, we can easily see that the convergence above is uniform. If we integrate term by term we obtain

$$-\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n},$$

In particular, we obtain the following elegant expression for ln 2 by substituting $x = \frac{1}{2}$:

$$\ln 2 = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \dots$$

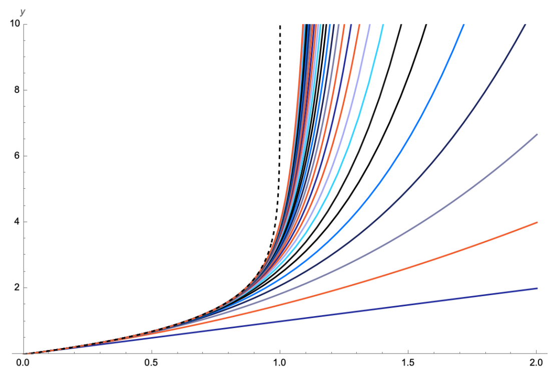


Fig. 8.12: The function $-\ln(1-x)$ (dashed) and some of its partial sum s_k $(1 \le k \le 30)$

Now, we discuss the case of derivatives. In this case, uniform convergence alone is not sufficient to interchange the limit and the derivative; a stronger assumption is required.

Theorem 8.23. *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of differentiable functions. Suppose there exists $c \in [a,b]$ such that $f_n(c)$ converges, and additionally, $f_n' \to g$ uniformly for some $g : [a,b] \to \mathbb{R}$. Then there exists a differentiable function $f : [a,b] \to \mathbb{R}$ such that $f_n \to f$ uniformly and $f' = g$. In other words,*

$$(\lim f_n)' = \lim f_n'.$$

Proof. We apply the Mean Value Theorem to the function $f_n - f_m$ on the interval $[c,x]$ to obtain $d \in (c,x)$, such that

$$f_n(x) - f_m(x) = f_n(c) - f_m(c) + (x-c)[f_n'(d) - f_m'(d)]. \tag{8.2}$$

Since f_n' converges uniformly, the sequence f_n satisfies Cauchy's criterion. Hence, $f_n \to f$ uniformly, for some $f : [a,b] \to \mathbb{R}$. We may now rewrite (8.2) using an arbitrary point $x_0 \in [a,b]$ instead of c. For $x \neq x_0$, we have

$$\frac{f_n(x) - f_n(x_0)}{x - x_0} - \frac{f_m(x) - f_m(x_0)}{x - x_0} = f_n'(d) - f_m'(d).$$

Define

$$q_n(x) = \frac{f_n(x) - f_n(x_0)}{x - x_0}, \quad x \neq x_0.$$

It follows that q_n is a Cauchy sequence, and thus converges uniformly on $[a, b] \setminus \{x_0\}$ to $\frac{f(x)-f(x_0)}{x-x_0}$. Using Theorem 8.14 we conclude that

$$f'(x_0) = \lim_{x \to x_0} \left[\lim_{n \to +\infty} q_n(x) \right] = \lim_{n \to +\infty} \left[\lim_{x \to x_0} q_n(x) \right] = g(x_0).$$

Since $x_0 \in [a, b]$ was arbitrary, we conclude that $f' = g$. □

Corollary 8.24. *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of differentiable functions. Suppose there exists $c \in [a, b]$ such that the series $\sum_{n=1}^{+\infty} f_n(c)$ converges, and additionally, the series $\sum_{n=1}^{+\infty} f_n'$ converges uniformly to some $g : [a, b] \to \mathbb{R}$. Then there exists a differentiable function $f : [a, b] \to \mathbb{R}$ such that $\sum_{n=1}^{+\infty} f_n = f$ uniformly and $f' = g$*

Example 8.25. Consider the series

$$\sum_{n=1}^{+\infty} \frac{\sin\left(\frac{x}{n}\right)}{n}.$$

The series clearly converges when $x = 0$. Moreover, the series formed by its derivatives $\sum_{n=1}^{+\infty} \frac{\cos\left(\frac{x}{n}\right)}{n^2}$ converges uniformly by the $M-test$. Therefore, $\sum_{n=1}^{+\infty} \frac{\sin\left(\frac{x}{n}\right)}{n}$ also converges uniformly on any closed interval containing zero.

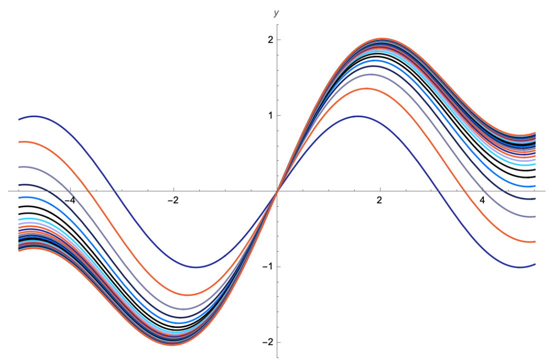


Fig. 8.13: Partial sums s_k $(1 \leq k \leq 30)$ of $\sum\limits_{n=1}^{+\infty} \frac{\sin\left(\frac{x}{n}\right)}{n}$

8.3 Power Series

In this section, we discuss series of functions of the form

$$\sum_{n=0}^{+\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \ldots,$$

which are commonly referred to as *power series*.

For simplicity, we will suppose $x_0 = 0$. This assumption does not affect the results that follow.

Example 8.26. Recall that by Example 3.56, the numerical series $\sum\limits_{n=0}^{+\infty} a^n$ converges if and only if $|a| < 1$. Therefore, the power series

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + \ldots,$$

converges uniformly in the open interval $(-1, 1)$, and diverges for $x \geq 1$. In fact, as previously discussed in Example 6.45, $\sum\limits_{n=0}^{+\infty} x^n = \frac{1}{1-x}$.

Example 8.27. As we saw in Example 6.44, the exponential function is analytic and admits a power series representation:

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$$

Notice that this is the same function discussed in Example 7.15.

Theorem 8.28. *If a power series* $\sum_{n=0}^{+\infty} a^n x^n$ *is convergent then exactly one of the following holds:*

a. The series converges only at $x = 0$*;*
b. The series converges for all $x \in \mathbb{R}$*;*
c. The series converges for all $x \in (-r, r)$ *and diverges for* $x \notin [-r, r]$*, where*

$$r = \frac{1}{\limsup \sqrt[n]{|a_n|}}$$

satisfies $0 < r < +\infty$*. At the endpoints* $x = \pm r$*, the series may converge or diverge.*

Proof. We analyze the sequence $\sqrt[n]{|a_n|}$. Suppose this sequence is unbounded. Then observe that

$$\sum_{n=0}^{+\infty} |a_n x^n| = \sum_{n=0}^{+\infty} \left(\sqrt[n]{|a_n|} |x| \right)^n .$$

It follows that if $x \neq 0$, the sequence $\sqrt[n]{|a_n|}|x|$ does not converge to zero. Therefore, in this case, the series $\sum_{n=0}^{+\infty} a^n x^n$ converges only at $x = 0$.

Now, suppose

$$\lim_{n \to +\infty} \sqrt[n]{|a_n|} = 0.$$

Applying the root test to the series $\sum_{n=0}^{+\infty} |a_n x^n|$, we conclude that it is absolutely convergent for all $x \in \mathbb{R}$, since

$$\lim_{n \to +\infty} \sqrt[n]{|a_n x^n|} = |x| \lim_{n \to +\infty} \sqrt[n]{|a_n|} = 0.$$

The only possibility left is

$$\limsup \sqrt[n]{|a_n|} = \frac{1}{r},$$

for some $r > 0$. Notice that

$$\lim_{n \to +\infty} \sqrt[n]{|a_n x^n|} = |x| \lim_{n \to +\infty} \sqrt[n]{|a_n|} = \frac{|x|}{r}.$$

Therefore, by applying the root test again we obtain that the series converges for $|x| < r$ and diverges for $|x| > r$. □

The number $r > 0$ described above is called the *radius of convergence* of the series $\sum_{n=0}^{+\infty} a^n x^n$. By convention, if the series converges only at zero, we set $r = 0$; and if it converges for all $x \in \mathbb{R}$, we set $r = +\infty$.

Corollary 8.29. *The series $\sum_{n=0}^{+\infty} a^n x^n$ converges uniformly in every closed interval contained in $(-r, r)$, where r is the radius of convergence of the series.*

Proof. This is an immediate consequence of the M-test. □

Notice that the Corollary does not say however, that $\sum_{n=0}^{+\infty} a^n x^n$ converges uniformly in the whole interval $(-r, r)$.

For Example, the series $\sum x^n$ can't converge uniformly in $(-1, 1)$, since this would imply convergence on the endpoints ± 1. On the other hand, if the series does converge at the endpoints $\pm r$, then the convergence is indeed uniform, as the follows Theorem shows.

Theorem 8.30. *(Abel) Let $0 < r < +\infty$ be the radius of convergence of the series $\sum_{n=0}^{+\infty} a^n x^n$. If $\sum_{n=0}^{+\infty} a^n r^n$ converges, then $\sum_{n=0}^{+\infty} a^n x^n$ is uniformly convergent on $[0, r]$. A similar result holds if $\sum_{n=0}^{+\infty} a^n (-r)^n$ converges. In particular, if the series converges at $\pm r$, then it converges uniformly on $[-r, r]$.*

Proof. By the Cauchy Criterion, given any $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n > n_0 \Rightarrow |a^{n+1} r^{n+1} + a^{n+2} r^{n+2} + \ldots + a^{n+m} r^{n+m}| < \epsilon \quad \text{for all } m \in \mathbb{N}$$

Define $y_m = a^{n+m} r^{n+m}$, and let $s_k = \sum_{m=1}^{k} y_m$ be its partial sum. Then for all $x \in [0, r]$:

$$\begin{aligned}
|a^{n+1} x^{n+1} + \ldots + a^{n+m} x^{n+m}| &= \left(\frac{x}{r}\right)^n \left| y_1 \left(\frac{x}{r}\right) + \ldots + y_m \left(\frac{x}{r}\right)^m \right| \\
&= \left(\frac{x}{r}\right)^n \left| s_1 \left(\frac{x}{r}\right) + (s_2 - s_1) \left(\frac{x}{r}\right)^2 + \ldots + (s_m - s_{m-1}) \left(\frac{x}{r}\right)^m \right| \\
&= \left(\frac{x}{r}\right)^n \left| s_1 \left(\left(\frac{x}{r}\right) - \left(\frac{x}{r}\right)^2\right) + \ldots + s_m \left(\frac{x}{r}\right)^m \right| \\
&< \left(\frac{x}{r}\right)^n \epsilon \left(\frac{x}{r}\right) \\
&\leq \epsilon
\end{aligned}$$

Therefore, the series $\sum_{n=0}^{+\infty} a^n x^n$ converges uniformly on $[0, r]$. □

Example 8.31. According to the Fundamental Theorem of Calculus, the function $\arctan : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ can be expressed as

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}\, dt.$$

On the other hand, observe that for every $t \in (-1, 1)$, the function $\frac{1}{1+t^2}$ admits the power series expansion

$$\frac{1}{1+t^2} = \sum_{n=0}^{+\infty} (-1)^n t^{2n},$$

where the convergence is uniform on every closed interval contained in $(-1, 1)$. By integrating the series term by term, we obtain

$$\arctan(x) = \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^{2n}\, dt = \sum_{n=0}^{+\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Applying Abel's Theorem, we conclude that this series converges uniformly on the closed interval $[-1, 1]$. In particular, evaluating at $x = 1$ yields the classical identity

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Theorem 8.32. *Let r be the radius of convergence of the power series $\sum\limits_{n=0}^{+\infty} a_n x^n$. Then for any $[a, b] \subseteq (-r, r)$:*

$$\int_a^b \sum_{n=0}^{+\infty} a_n x^n\, dx = \sum_{n=0}^{+\infty} \frac{a_n}{n+1}(b^{n+1} - a^{n+1})$$

Proof. This follows directly from Corollary 8.21. □

Theorem 8.33. *Let r be the radius of convergence of the power series $\sum\limits_{n=0}^{+\infty} a_n x^n$. Define the function $f : (-r, r) \to \mathbb{R}$ by*

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n.$$

Then f is differentiable with its derivative given by

$$f'(x) = \sum_{n=1}^{+\infty} a_n n x^{n-1}.$$

Moreover, the radius of convergence of the power series defining f' is also r.

Proof. Let $\bar{r}$ be the radius of convergence of $\sum_{n=1}^{+\infty} a_n n x^{n-1}$. Observe that $\bar{r}$ is also the radius of convergence of $\sum_{n=1}^{+\infty} a_n n x^n$. We analyze the convergence of this latter series. Suppose $\rho > 0$ satisfies $0 < \rho < r$. Choose $c > 0$, such that

$$0 < \rho < c < r.$$

It follows that for n sufficiently large,

$$\sqrt[n]{|a_n|} < \frac{1}{c}.$$

On the other hand, for n sufficiently large, we clearly have

$$\sqrt[n]{n} < \frac{1}{\rho}.$$

Combining these two estimates yields

$$\sqrt[n]{|na_n|} < \frac{1}{\rho},$$

and hence, $0 < \rho < \bar{r}$. We conclude that $0 < \rho < r \Rightarrow 0 < \rho < \bar{r}$, this can only occur if $r = \bar{r}$. By applying Corollary 8.24, we have $f'(x) = \sum_{n=1}^{+\infty} a_n n x^{n-1}$. □

Corollary 8.34. *Let r be the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n x^n$. Define the function $f : (-r, r) \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{+\infty} a_n x^n$. Then f is of class C^∞ (it has all of its derivatives), and moreover,*

$$a_n = \frac{f^n(0)}{n!}.$$

In other words, f is analytic with power series expression given by its Taylor series around zero.

Corollary 8.35. *Let $X \subseteq \mathbb{R}$ be a set with the property that $0 \in X'$. Suppose $\sum_{n=0}^{+\infty} a_n x^n$ and $\sum_{n=0}^{+\infty} b_n x^n$ are two convergent power series on $(-r, r)$, such that $\sum_{n=0}^{+\infty} a_n x^n = \sum_{n=0}^{+\infty} b_n x^n$ on X. Then $a_n = b_n$ for $n \in \mathbb{N}$.*

Proof. By hypothesis, the functions $f := \sum_{n=0}^{+\infty} a_n x^n$ and $g := \sum_{n=0}^{+\infty} b_n x^n$ satisfies

$$f^n(0) = g^n(0).$$

Hence, $a_n = \frac{f^n(0)}{n!} = \frac{g^n(0)}{n!} = b_n$. □

Example 8.36. (Binomial Series) For $\alpha \in \mathbb{R}$ and $n \in \mathbb{N} \cup \{0\}$, define

$$\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{k!} \quad \text{if } n \neq 0,$$

and $\binom{\alpha}{0} = 1$. We analyze the convergence of the power series

$$\sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n.$$

Observe that

$$\lim_{n\to+\infty} \left| \frac{\binom{\alpha}{n+1}}{\binom{\alpha}{n}} \right| = \left| \frac{n-\alpha}{n+1} \right| = 1.$$

Therefore, by using the ratio test we conclude that the radius of convergence of this series is 1, i.e., the series converges for $|x| < 1$, and diverges if $|x| > 1$.

For $x \in (-1, 1)$, define $f(x) = \sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n$. A quick computation shows that f satisfies

$$(1+x)f'(x) = \alpha f(x).$$

Now, define $g(x) = \frac{f(x)}{(1+x)^\alpha}$, then

$$g'(x) = \frac{f'(x)(1+x)^\alpha - f(x)\alpha(1+x)^{\alpha-1}}{(1+x)^{2\alpha}} = 0.$$

Hence, g is constant, but since $g(0) = 1$, we obtain

$$\sum_{n=0}^{+\infty} \binom{\alpha}{n} x^n = (1+x)^\alpha.$$

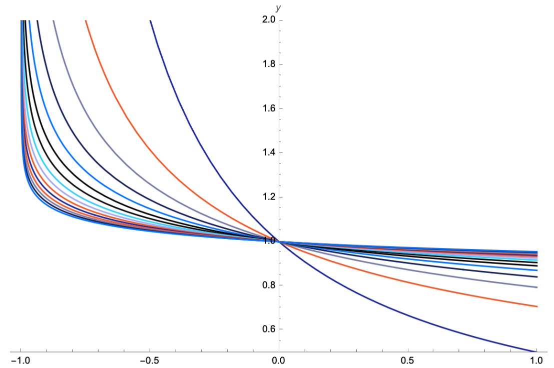


Fig. 8.14: The sequence $f_n(x) = (1+x)^{-\frac{1}{n}}$ for $(1 \leq n \leq 20)$

Example 8.37. (The Basel Problem) The following problem was first proposed in 1650 by Italian Mathematician Pietro Mengoli:

"What is the precise value of the series $\sum\limits_{n=1}^{+\infty} \frac{1}{n^2}$?"

We already know that this series converges, namely, due the p-series test. However, it took several decades before a closed-form expression for the sum was discovered.

In 1734, the Swiss mathematician Leonhard Euler showed that the series converges to $\frac{\pi^2}{6}$. Euler's proof relied on techniques involving infinite products. Here, we present an alternative solution due to B. Choe, published in 1987.

Recall that

$$\arcsin(x) = \int_0^x \frac{1}{\sqrt{1-t^2}}\,dt$$

Using Example 8.36 with $\alpha = -\frac{1}{2}$, we obtain for any $x \in [-1, 1]$:

$$\arcsin(x) = \sum_{n=0}^{+\infty} \frac{(2n-1)!!}{(2n)!!}\frac{x^{2n+1}}{2n+1},$$

where the double factorial is defined by $n!! = n(n-2)(n-4)(n-6)\ldots$.

Now, set $x = \sin t$. Substituting into the series, we find

$$t = \sum_{n=0}^{+\infty} \frac{(2n-1)!!}{(2n)!!}\frac{(\sin t)^{2n+1}}{2n+1}$$

for any $t \in [0, 2\pi]$. Integrating both sides from 0 to π, we have

$$\frac{\pi^2}{8} = \sum_{n=0}^{+\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \int_0^{\pi} (\sin t)^{2n+1}\, dt$$
$$= \sum_{n=0}^{+\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \frac{(2n)!!}{(2n+1)!!}$$
$$= \sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2}.$$

On the other hand, observe that

$$\sum_{n=0}^{+\infty} \frac{1}{(2n+1)^2} = \sum_{n=1}^{+\infty} \frac{1}{n^2} - \sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{3}{4} \sum_{n=1}^{+\infty} \frac{1}{n^2}.$$

It follows that

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercises

1. Let $f_n : [0, +\infty) \to \mathbb{R}$ be a sequence of functions defined by

$$f_n(x) = \frac{x^n}{x^n + 1}.$$

Show that $f_n \to f$ pointwise, but not uniformly, for some function f. Find an explicit expression for the limiting function f.

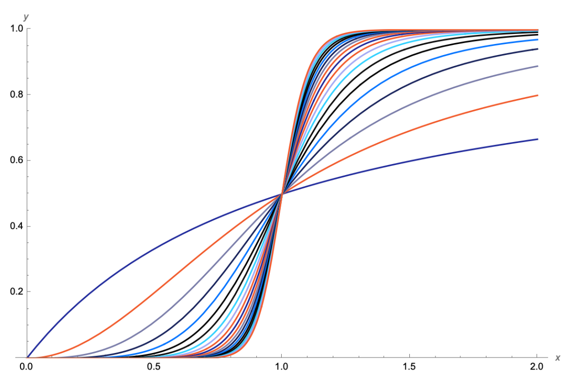


Fig. 8.15: The sequence $f_n(x) = \frac{x^n}{x^n+1}$ for $(1 \leq n \leq 20)$

2. Find the radius of convergence of the series $\sum_{n=1}^{+\infty} x^n(1-x^n)$.
3. Show that $\sum_{n=1}^{+\infty} |f_n| < \infty \Rightarrow \sum_{n=1}^{+\infty} f_n < \infty$.
4. Consider the series
$$\sum_{n=1}^{+\infty} \frac{1}{1+n^2x}.$$
For what values of x does this series converge uniformly?
5. Let $f_n : [0,1] \to \mathbb{R}$ be a sequence defined by $f_n(x) = \frac{\sin nx}{\sqrt{n}}$. Show that $f_n \to 0$ uniformly, but f_n' does not converge anywhere in $[0,1]$.
6. Consider the sequence $f_n(x) = x + \frac{x^n}{n}$ on the interval $[0,1]$. Show that f_n converges uniformly to some function g, and moreover, show that f_n' converges pointwise but its limit is not g'.
7. Prove that if $f_n \to f$ uniformly in a dense subset $D \subseteq X$, then $f_n \to f$ uniformly in X.
8. Find the radius of convergence of the series $\sum_{n=1}^{+\infty} n^{\frac{\ln n}{n}} x^n$.
9. Let a_n denotes the Fibonacci sequence given by $a_0 = a_1 = 1$ and
$$a_{n+1} = a_n + a_{n-1}.$$
Find the radius of convergence of the power series $\sum_{n=0}^{+\infty} a_n x^n$.
10. Show that if $f_n \to f$ uniformly on $X \subseteq \mathbb{R}$, and each f_n is uniformly continuous then f is uniformly continuous in X.
11. Prove that a sequence of polynomials p_n cannot converge uniformly to $\frac{1}{x}$ on the interval $(0,1)$.
12. Give an example of a sequence of function $f_n : [a,b] \to \mathbb{R}$ that converges uniformly on (a,b), but does not converge at the endpoints.
13. Given $\epsilon > 0$, show that the series $\sum_{n=1}^{+\infty} \frac{\sin nx}{n}$ converges uniformly on $[\epsilon, 2\pi - \epsilon]$.
14. Show that the series $\sum_{n=0}^{+\infty} \frac{x^n}{n!}$ does not converge uniformly on the entire real line $(-\infty, \infty)$.

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