Exercises

1. Let A, B, X be sets with the following properties:

   $A \subseteq X$ and $B \subseteq X$

   For any set Y if $A \subseteq Y$ and $B \subseteq Y$ then $X \subseteq Y$.

   Show that $X = A \cup B$.

   Solution. Obviously $A \cup B \subseteq X$. Take $Y = A \cup B$, then it follows that $X \subseteq A \cup B$ and hence $X = A \cup B$. □

5. Given two sets A, B we define the *symmetric difference* $A \Delta B$ by

   $$A \Delta B = (A - B) \cup (B - A).$$

   Prove that if $A \Delta B = A \Delta C$, then $B = C$.

   Solution. We have

   $$\begin{aligned}(A\Delta B)\Delta A &= (A\Delta B - A) \cup (A - A\Delta B)\\ &= (B - A) \cup (A - (A - B))\\ &= B\end{aligned}$$

   A similarly, $(A\Delta C)\Delta A = C$. Hence, $B = C$. □

7. Show that a function $f : A \to B$ is injective if and only if $f(A - X) = f(A) - f(X)$ for every $X \subseteq A$.

   Solution. Suppose f injective. Take $y \in f(A - X)$,i.e. $y = f(a)$ for $a \in A$, but $a \notin X$, by the injectivity of f, $y \in f(A) - f(X)$. Now, suppose $y \in f(A) - f(X)$, then $y = f(a)$ but $y \neq f(x)$ for any $x \in X$, again, by the injectivity of f, $a \notin X$, hence $y \in f(A - X)$.

   Conversely, suppose $f(A - X) = f(A) - f(X)$. Set $X = \{a\}$ for $a \in A$, then $f(A - \{a\}) = f(A) - f(a)$. In particulat, if $b \neq a$ then $f(b) \neq f(a)$. □

12. Given two natural numbers $a, b \in \mathbb{N}$, prove that there is a natural number $m \in \mathbb{N}$ such that $m \cdot a > b$.

    Solution. Suppose not, then the set $X = \{m \cdot a\, ;\, m \in \mathbb{N}\}$ would be bounded, hence finite; a contradiction. □

14. A number $a \in \mathbb{N}$ is called **predecessor** of $b \in \mathbb{N}$ if $a < b$ and there is no $c \in \mathbb{N}$ such that $a < c < b$. Prove that every number, except 1, has a predecessor.

Solution. Since 1 is the smallest natural number, there is no number $a \in \mathbb{N}$ such that $a < 1$. Let $n > 1$, then $n = s(m) = m + 1$. We claim m is its predecessor. Indeed, $m < n$ and suppose there is $c \in \mathbb{N}$ such that $m < c < n$. Since the successor function is surjective, we have $c = p+1, m = q+1$ for some $p \in \mathbb{N}$. Then $q < p < n$. Iterating this procedure, we obtain a number $k \in \mathbb{N}$ with the property that $1 < k < 2$, since $k > 1$, it is the successor of a number $q \in \mathbb{N}$, but then $q+1 < 2$ or $q < 1$, a contradiction. □

19. Give an example of a surjective function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $f^{-1}(n)$ is infinite.

    Solution. We proved in class that $\mathbb{N} \times \mathbb{N}$ is countable, so there is a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Consider the projection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $\pi(x, y) = x$. Then $\pi \circ f$ is an example. Another approach that we mentioned in class is partition $\mathbb{N}$ into a countable union of disjoint subsets. □

21. Show that if A is countably infinite then $\mathcal{P}(A)$ is uncountable.

    Solution. Notice that $\mathcal{P}(A) = \mathcal{F}(A; \{0, 1\})$. Cantor's theorem gives that $\mathcal{P}(A)$ is uncountable. □

27. (Cantor-Bernstein-Schroder theorem) Given sets A and B, let $f : A \to B$ and $g : B \to A$ be injective functions. Show that there is a bijection $h : A \to B$.

    Solution. Notice that given $x \in A$, after successive applications of f and g we produce a path that either lands back at x, or doesn't. In the former case, set $h(x) = f(x)$. If we don't land back at x, we have an infinite path starting at x or containing x. If the path starts in A, set $h(x) = f(x)$, whereas if the path starts in B, set $h(x) = g^{-1}(x)$. If the path if infinite, contains x but is not cyclic, set $h(x) = f(x)$. The functions h is a bijection by construction. □