Exercises

11. Let $\mathcal{F}(X;Y)$ denote the set of all functions with domain X and codomain Y. Given the sets A, B, C, show that there is a bijection

$$\mathcal{F}(A \times B; C) \to \mathcal{F}(A; \mathcal{F}(B; C)).$$

Solution. Define $h : \mathcal{F}(A \times B; C) \to \mathcal{F}(A; \mathcal{F}(B; C))$ by

$$h(f) = \tilde{f} : x \mapsto f_x(y)$$

where $f_x(y) = f(x, y)$. We claim h is bijective.

Suppose $h(f) = h(g)$, then

$$\tilde{f}(x) = \tilde{g}(x)$$

for every $x \in A$. But then $f_x(y) = g_x(y)$ for every $x \in A, y \in B$. Hence, $f(x, y) = g(x, y)$ and h is injective.

Now, let $k : A \to \mathcal{F}(B; C)$ be given. Consider the function $j : A \times B \to C$ given by

$$j(x, y) = k(x)(y).$$

Then $h(j) = k$ by definition of h. It follows that h is surjective, and hence bijective. □

12 Let $a \in \mathbb{N}$. If the set $X \subseteq \mathbb{N}$ has the following property: $a \in X$ and $n \in X \Rightarrow n+1 \in X$. Then X contains all natural numbers greater than or equal to a.

Solution. Indeed, notice that for any $m > a$, we have $m = a + (m - a)$. So it suffices to prove that numbers of the form $a + n \in X$ for every $n \in \mathbb{N}$. We argue by induction. The case $n = 1$ is trivial, suppose the result valid for n. But since $n \in X \Rightarrow n+1 \in X$, we have $(a + n) + 1 \in X$, hence $a + n + 1 \in X$, and the conclusion follows. □

16. Using strong induction show that the decomposition of any number in prime factors is unique.

Solution. Let n be given and suppose the result valid for every $m < n$. If n is prime there is nothing to prove, if n is composite say $n = pq$, then p and q have a unique prime decomposition by hypothesis. Hence, n has a unique prime decomposition. By strong induction, it follows that every number has a unique prime decomposition. □

18. Let X be a finite set. Show that a function $f : X \to X$ is injective $\iff$ is surjective.

Solution. Since X is finite, say $|X| = n$. If f is injective, the image of f has at least n elements, thus it has exactly n and f is surjective. Conversely, if f is surjective then its image has exactly n elements, if f is not injective then its image has less than n elements and it follows that f has to be injective. □

21. Show that if A is countably infinite then $\mathcal{P}(A)$ is uncountable.

Solution. By Cantor's theorem, the cardinality of A is sctrictly less then that of $\mathcal{P}(A)$, so there can be no injective function from A to $\mathcal{P}(A)$. $\square$

22. Let $f : X \to X$ be injective but not surjective. If $x \in X - f(X)$, show that $x, f(x), f(f(x)), \ldots$ are pairwise distinct.

    Solution. We proof by induction over the number of times f is composed. The case $x, f(x)$ is trivial since $x \in X - f(X)$. Suppose the result valid for n compositions of f and apply f to each element $x, f(x), f(f(x)), \ldots, f^{(n)}(x)$. Then the elements $f(x), f(f(x)), \ldots, f^{(n+1)}(x)$ are all distinct since f is injective. They can't be x either by the inductive hypothesis and the fact that $x \in X - f(X)$ implies $f^{n+1}(x) \neq x$. $\square$

23. Let X be an infinite set e Y a finite set. Show that there is a surjective function $f : X \to Y$ and an injective function $g : Y \to X$.

    Solution. Suppose $Y = \{y_1, y_2, \ldots, y_n\}$. Choose n distinct elements of X, say $x_1, \ldots, x_n$. The function $f(x_i) = y_i$ for $i = 1, \ldots, n$ and $f(x) = y_1$ otherwise, is surjective. Similarly, the function $g(y_i) = x_i$ defines an injective function. $\square$

28. Given a sequence of sets $A_1, A_2, A_3, \ldots$, we define the *limit superior* as the set

$$\limsup A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} A_i \right).$$

Similarly, the *limit inferior* is the set

$$\liminf A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{i=n}^{\infty} A_i \right).$$

a. Show that $\limsup A_n$ is the set of elements that belong to A_i for infinitely many values of i. Similarly, show that $\liminf A_n$ is the set of elements that belong to A_i for every value of i, except possibly, for a finite number of values of i.

Solution. We prove the result for $\limsup A_n$, the other statement is analogous. By definition, if $x \in \bigcap_{n=1}^{\infty} \left(\bigcup_{i=n}^{\infty} A_i \right)$, then $x \in \bigcup_{i=n}^{\infty} A_i$ for every $n \in \mathbb{N}$. If x was in finitely many A_i, say in $A_{n_1}, A_{n_2}, \ldots, A_{n_k}$ then it wouldn't be in $x \in \bigcup_{i=n_k+1}^{\infty} A_i$. Therefore, x belong to $A_i$for infinitely many values of i. $\square$

b. Conclude that $\liminf A_n \subseteq \limsup A_n$.

Solution. If x belong to A_i for every value of i, except possibly, for a finite number of values of i then it obviously belong to A_i for infinitely many values of i. $\square$

c. Show that if $A_n \subseteq A_{n+1}$ for every n then $\liminf A_n = \limsup A_n = \bigcup_{n=1}^{\infty} A_n$.

Solution. Notice that in this case $\bigcap_{i=n}^{\infty} A_i = A_n$, so $\liminf A_n = \bigcup_{n=1}^{\infty} A_n$. By the item above, it suffices to prove that $\limsup A_n \subseteq \bigcup_{n=1}^{\infty} A_n$, but this follows from the fact that $\limsup A_n$ is the set of elements that belong to A_i for infinitely many values of i. □

d. Show that if $A_{n+1} \subseteq A_n$ for every n then $\liminf A_n = \limsup A_n = \bigcap_{n=1}^{\infty} A_n$.

Solution. Proof is analogous to the that of item c. □

e. Give an example of sequence such that $\liminf A_n \neq \limsup A_n$.

Solution. $A_n = \{(-1)^n\}$. □