Exercises

12. Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$. Show that x_n is bounded and find $\lim x_n$.

    Solution. Suppose x_n converges, say $\lim x_n = L$, then taking the limit on $x_{n+1} = 1 + \sqrt{x_n}$ we obtain:
$$L = 1 + \sqrt{L}$$
    Thus, either $L = \frac{3-\sqrt{5}}{2}$ or $L = \frac{3+\sqrt{5}}{2}$.

    We argue by induction that x_n is increasing and bounded by $L = \frac{3+\sqrt{5}}{2}$. The case $n = 1$ is clear. Suppose x_n increases at k, we proof it also increases at $k + 1$. Indeed,
$$x_k < x_{k+1} \Rightarrow 1 + \sqrt{x_k} < 1 + \sqrt{x_{k+1}} \Rightarrow x_{k+1} < x_{k+2},$$
    hence, x_n is increasing. Similarly, if $x_n < L = \frac{3+\sqrt{5}}{2}$ then $x_{n+1} = 1+\sqrt{x_n} < 1+\sqrt{L} = L$, and x_n is bounded by L. Since x_n is bounded and increasing it converges, since it can't converge to $\frac{3-\sqrt{5}}{2}$ (due to its monotonicity), we must have $\lim x_n = \frac{3+\sqrt{5}}{2}$. $\square$

14. Let $y_n > 0$ for every $n \in \mathbb{N}$, such that $\sum y_n = +\infty$. If x_n is a sequence such that $\lim \frac{x_n}{y_n} = a$, show that $\lim \frac{x_1+\ldots+x_n}{y_1+\ldots+y_n} = a$.

    Solution. By hypothesis, there exists $n_0 \in \mathbb{N}$ such that
$$n > n_0 \Rightarrow \frac{x_n}{y_n} \in (a - \epsilon, a + \epsilon)$$
    That is,
$$(a - \epsilon)y_n < x_n < (a + \epsilon)y_n$$
    Summing from $n = n_0 + 1$ to $n = k$, we have
$$\sum_{n=n_0+1}^{k} (a - \epsilon)y_n < \sum_{n=n_0+1}^{k} x_n < \sum_{n=n_0+1}^{k} (a + \epsilon)y_n$$
    adding the remainder terms,we obtain
$$\sum_{n=1}^{n_0} x_n + \sum_{n=n_0+1}^{k} (a - \epsilon)y_n < \sum_{n=1}^{n_0} x_n + \sum_{n=n_0+1}^{k} x_n < \sum_{n=1}^{n_0} x_n + \sum_{n=n_0+1}^{k} (a + \epsilon)y_n$$
    Finally, diving by $\sum_{n=1}^{k} y_n$ and noticing that $\sum_{n=1}^{k} y_n \to +\infty$, by hypothesis, we obtain for k sufficiently large
$$a - \epsilon < \frac{x_1 + \ldots + x_k}{y_1 + \ldots + y_k} < a + \epsilon$$
    We conclude that $\lim \frac{x_1+\ldots+x_n}{y_1+\ldots+y_n} = a$. $\square$

15. (Stolz-Cesaro Theorem) Let y_n be an increasing sequence and $\lim y_n = +\infty$. Show that

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = a \Rightarrow \lim \frac{x_n}{y_n} = a.$$

Hint: Use the exercise above.

Solution. Let $u_n = x_{n+1} - x_n$ and $v_n = y_{n+1} - y_n$ then by the exercise above

$$a = \lim \frac{u_1 + \ldots + u_n}{v_1 + \ldots + v_n} = \lim \frac{x_n - x_1}{y_n - y_1} = \lim \frac{\frac{x_n}{y_n} - \frac{x_1}{y_n}}{1 - \frac{y_1}{y_n}} = \lim \frac{x_n}{y_n}$$

□

17. Show that for every $n \in \mathbb{N}$, $0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) < \frac{1}{n!n}$. Conclude that $e \notin \mathbb{Q}$.

Solution. We first show that $e \notin \mathbb{Q}$. Otherwise, if $e = \frac{p}{q}$ with p, q coprime, then

$$0 < p - q\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) < \frac{q}{n!n}$$

Choosing $n = q$:

$$0 < q!p - q!q\left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{q!}\right) < 1$$

A contradiction, since the middle number is a integer, hence can't be between 0 and 1. Now, by Example 3.28, we know that $e = \lim(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!})$, hence

$$\begin{aligned} e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \ldots \\ &\leq \frac{1}{n!}\left(\frac{1}{n+1} + \frac{1}{(n+1)(n+1)} + \ldots\right) \\ &= \frac{1}{n!} \cdot \frac{1}{n} \end{aligned}$$

□

19. Suppose the sequence x_n satisfies $n! = n^n e^{-n} x_n$. Show that $\lim \sqrt[n]{x_n} = 1$.

Solution. Recall that $\lim \frac{x_{n+1}}{x_n} = L \Rightarrow \lim \sqrt[n]{x_n} = L$. It suffices to show that $\lim \frac{x_{n+1}}{x_n} = 1$. We have

$$\lim \frac{x_{n+1}}{x_n} = \lim \frac{(n+1)!e^{n+1}}{(n+1)^{n+1}} \cdot \frac{n^n}{n!e^n} = e \lim \left(\frac{n}{n+1}\right)^n = e\frac{1}{e} = 1.$$

□