A nonlinear problem related to optimal insulation

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Abstract

We study a generalization of the classical optimal insulation problem by replacing the standard Laplacian with the *p-Laplacian*, leading to a *p-Poisson equation* with Robin boundary conditions. We also investigate the corresponding eigenvalue problem. This work extends some results from [6, 14] to the nonlinear setting.

Keywords: p-Laplacian, Robin boundary conditions, optimal insulation

MSC Classification: 35J25 , 35J92 , 49R05

1 Introduction

In this work, we investigate a generalization of the classical optimal insulation problem. Broadly speaking, the classical problem seeks the optimal distribution of an insulating material around a fixed thermally conducting body.

Mathematically, the fixed conducting body is represented by an open, bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and the insulating material is modeled by a set $\Sigma_\varepsilon \subset \mathbb{R}^n$ defined by

$$\Sigma_\varepsilon = \{\sigma + t\nu(\sigma)\,;\, \sigma \in \partial\Omega,\, 0 \leq t < \varepsilon h(\sigma)\},$$

where $\nu(\sigma)$ denotes the unit outward normal (which is well-defined since Ω is assumed to have a Lipschitz boundary), and $h : \partial\Omega \to \mathbb{R}$ is a bounded positive Lipschitz function.

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Set $\Omega_\varepsilon = \Omega \cup \Sigma_\varepsilon$. If $u(x)$ denotes the temperature at a point $x \in \Omega_\varepsilon$, and f is a given source function, then $u \in H_0^1(\Omega_\varepsilon)$ minimizes the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx + \frac{\varepsilon}{2}\int_{\Sigma_\varepsilon} |Du|^2 \, dx - \int_\Omega fu\, dx. \tag{1}$$

Equivalently, u is a solution to the corresponding Euler–Lagrange equation:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ -\Delta u = 0 & \text{in } \Sigma_\varepsilon, \\ u = 0 & \text{on } \partial\Omega_\varepsilon, \\ \frac{\partial u^-}{\partial \nu} = \varepsilon \frac{\partial u^+}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The optimal insulation problem consists in studying the behavior of the solution u as $\varepsilon \to 0$. In this context, the notion of Γ-convergence in the L^2 topology is particularly well-suited for analyzing the limiting behavior of the functionals F_ε as $\varepsilon \to 0$.

In [8, 1], the following result is proved:

Theorem. *The functionals F_ε defined by (1) Γ-converge in the L^2 topology to the functional*

$$F(u) = \frac{1}{2}\int_\Omega |Du|^2 \, dx + \frac{\varepsilon}{2}\int_{\partial\Omega} \frac{u^2}{h} \, d\mathcal{H}^{N-1} - \int_\Omega fu\, dx.$$

Consequently, the unique minimizer of F satisfies the Euler–Lagrange equation:

$$\begin{cases} -\Delta u = f & \textit{in } \Omega, \\ h\frac{\partial u}{\partial \nu} + u = 0 & \textit{on } \partial\Omega. \end{cases} \tag{3}$$

In this work, we plan to discuss a generalized version of the functional F, namely:

$$F_h(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx + \frac{1}{p}\int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{N-1} - \int_\Omega fu$$

where $1 < p < N$, and $h : \partial\Omega \to \mathbb{R}$ is a Lipschitz function satisfying for every $\sigma \in \partial\Omega$:

$$0 < a \le h(\sigma) \le b,$$

for some $a, b > 0$. The associated Euler-Lagrange equation is given by

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ h^{p-1}|Du|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

The functional F_h is also related, in a sense, to the optimal insulation problem, as it arises as the Γ-limit of the family of functionals (see [1] for the proof) given by

$$F_h^\varepsilon(u) = \frac{1}{p}\int_\Omega |Du|^p \, dx + \frac{\epsilon}{p}\int_{\Sigma_\varepsilon} |Du|^p \, dx - \int_\Omega fu\, dx,$$

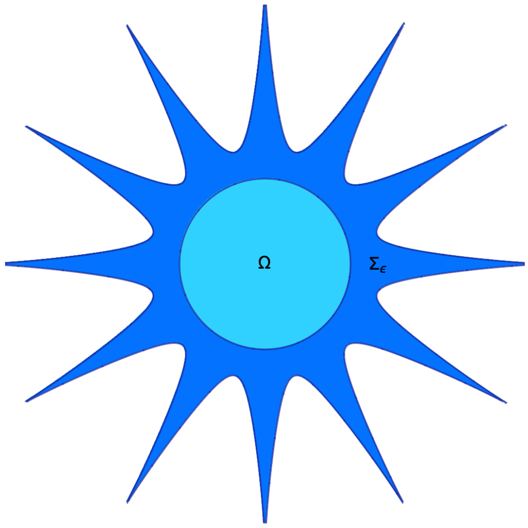

Fig. 1 Representation of disk $\Omega \subseteq \mathbb{R}^2$ with insulator described by the set Σ_ϵ. The function h describing the contour is not optimal for F_h if $p = 2$.

which coincides with (1) when $p = 2$.

When $p \neq 2$, the p-Laplacian still appears in various physical models, such as image denoising [3], sandpile modeling [13], and modeling of non-Newtonian fluids [10]. However, it is no longer directly related to heat transfer. Nonetheless, the mathematical analysis of F_h remains of interest even in the case $p \neq 2$.

Our first result addresses the following question:

Question: What is the optimal pair (u, h) that minimizes the value of $F_h(u)$, where $u \in W^{1,p}(\Omega)$ and h has fixed content, i.e., $\int_{\partial\Omega} h \, d\mathcal{H}^{N-1} = m$?

In other words, we aim to analyze the following double minimization problem:

$$\min_{h \in \mathcal{H}_m} \min_{u \in W^{1,p}(\Omega)} \{F_h(u)\}, \tag{5}$$

where

$$\mathcal{H}_m = \left\{ h : \partial\Omega \to \mathbb{R} \text{ measurable, } h \geq 0, \int_{\partial\Omega} h \, d\mathcal{H}^{d-1} = m \right\}.$$

In the case $p = 2$, this question was addressed in [6]. The present paper can be seen as a natural generalization of that work to the case $p \neq 2$. In this setting, the problem becomes nonlinear, and certain adjustments are required due to the presence of the nonlinearity $|Du|^{p-2}$ in the definition of the p-Laplacian.

In the discussion above, the set Ω is assumed to be fixed. It is natural to ask how the minimum of F_h behaves when both h and Ω are allowed to vary in an appropriate sense. Our second result below addresses this scenario. More precisely, we will show

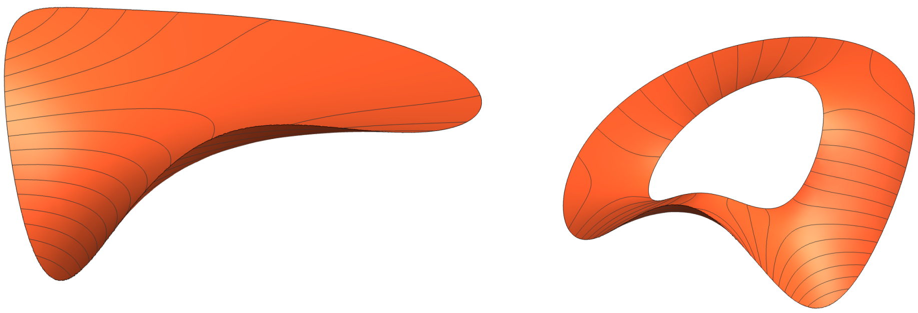

Fig. 2 Graph of the solution $u(x, y)$ to problem (6) with $p = 2$ and $h(x) = e^{x-y}$. Left: $\Omega = B_1 \subset \mathbb{R}^2$; Right: $\Omega = B_1 \setminus \overline{B_{1/2}}$.

that among all Lipschitz domains with fixed volume, the ball minimizes the value of F_h. This result constitutes a type of isoperimetric inequality and extends the work of [14], where the linear case is considered.

In the last section of this paper, we study the eigenvalue problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ h^{p-1}|Du|^{p-2}\frac{\partial u}{\partial \nu} + |u|^{p-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{6}$$

Using the direct method in the calculus of variations, it is straightforward to show that (6) admits a solution $u \in W^{1,p}(\Omega)$. Specifically, define the functional

$$J_h(u) = \frac{\displaystyle\int_\Omega |\nabla u|^p \; dx + \int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega |u|^p \, dx},$$

then any minimizer of

$$\min \left\{ J_h(u) : u \in W^{1,p}(\Omega), \; u \neq 0 \right\}$$

solves (6).

Our second result concerns the solution of the double minimization problem:

$$\min_{h \in \mathcal{H}_m} \; \min_{u \in W^{1,p}(\Omega)} J_h(u), \tag{7}$$

and our final result is the analysis of the same double minimization problem when we also allow the Lipschitz domain $\Omega \subset \mathbb{R}^n$ to vary. That is, we consider the following minimization problem:

$$\min_{|\Omega| \le 1} \; \min_{h \in \mathcal{H}_m} \; \min_{u \in W^{1,p}(\Omega)} J_h(u). \tag{8}$$

Notation & Assumptions

- $\Omega \subseteq \mathbb{R}^N$ is a bounded set with Lipschitz boundary.
- The space $W^{1,p}(\Omega)$ denotes the usual Sobolev space which is the closure of $\mathcal{C}_0^\infty(\Omega)$, smooth functions with compact support using the Sobolev norm.
- For $q > 1$, q' denotes the Holder conjugate, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$, and q^* denotes the Sobolev conjugate, defined by $q^* = \frac{qN}{N-q} > q$.
- The letter C will always denote a positive constant which may vary from place to place.
- The Lebesgue measure of a set $A \subseteq \mathbb{R}^n$ is denoted by $|A|$.
- The $N-1$ Hausdorff measure of a set $A \subseteq \mathbb{R}^n$ is denoted by $\mathcal{H}^{N-1}(A)$.
- The symbol $\rightharpoonup$ denotes weak convergence.

2 Solution to the double minimization problem (5)

The following lemma will be used in the proof below.

Lemma 1. *(Poincaré inequality) Let $u \in W^{1,p}(\Omega)$. Then there exists a constant $C > 0$ such that*

$$\int_\Omega |u|^p \, dx \leq C \left[\int_\Omega |Du|^p \, dx + \left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p \right]. \tag{9}$$

Proof. Suppose (9) is false. Then for there is a sequence $u_n \in W^{1,p}(\Omega)$ such that

$$\int_\Omega |u_n|^p \, dx > n \left[\int_\Omega |Du_n|^p \, dx + \left(\int_{\partial\Omega} |u_n| \, d\mathcal{H}^{N-1} \right)^p \right].$$

In particular,

$$\int_\Omega |Du_n|^p \, dx + \left(\int_{\partial\Omega} |u_n| \, d\mathcal{H}^{N-1} \right)^p \to 0$$

when $n \to +\infty$. Without loss of generality we may assume

$$\int_\Omega |u_n|^p \, dx = 1.$$

Hence, up to a subsequence, $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, with $\int_\Omega |u|^p \, dx = 1$. But since

$$\int_\Omega |Du_n|^p \, dx \to 0 \text{ and } \left(\int_{\partial\Omega} |u_n| \, d\mathcal{H}^{N-1} \right)^p \to 0$$

one must have $Du_n \to Du = 0$ strongly in $L^p(\Omega)$, and $u_n \to u = 0$ strongly in $L^p(\partial\Omega)$. Hence $u \equiv 0$, a contradiction. □

Theorem 2. *If Ω is connected, the minimization problem (5) admits a unique solution. In particular, if $\Omega = B_R$ and $f = 1$, then the optimal solution $h(x)$ is constant, given by*

$$h(x) = \frac{m}{N\omega_N R^{N-1}}.$$

and does not depend on p.

Proof. Note that given $u \in L^p(\partial\Omega)$ with $u \not\equiv 0$, the minimization problem

$$\min \left\{ \int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{N-1} \;:\; h \in \mathcal{H}_m \right\}$$

admits a unique solution. Indeed, define

$$\hat{h} = m \frac{|u|}{\left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)}.$$

Then, by Hölder's inequality,

$$\left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p \leq \left(\int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{N-1} \right) \left(\int_{\partial\Omega} h \, d\mathcal{H}^{N-1} \right)^{p-1},$$

which implies

$$\int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{N-1} \geq \frac{1}{m^{p-1}} \left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p = \int_{\partial\Omega} \frac{|u|^p}{\hat{h}^{p-1}} \, d\mathcal{H}^{N-1}.$$

Thus, $\hat{h}$ is a minimizer. Uniqueness follows from the strict convexity of the functional $G(h) = \int_{\partial\Omega} \frac{|u|^p}{h^{p-1}} \, d\mathcal{H}^{N-1}$ and the fact that Ω is connected.

It follows that the minimization problem (5) is equivalent to

$$\min \left\{ \frac{1}{p} \int_\Omega |Du|^p \, dx + \frac{1}{m^{p-1}p} \left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p - \int_\Omega fu \;:\; u \in W^{1,p}(\Omega) \right\}. \tag{10}$$

Notice that the functional

$$R(u) = \frac{1}{p} \int_\Omega |Du|^p \, dx + \frac{1}{m^{p-1}p} \left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p - \int_\Omega fu$$

is coercive by Lemma 1, albeit not differentiable. Additionally, the second and third terms are trivially convex. A simple computation shows that $\int_\Omega |Du|^p \, dx$ is strictly convex. It follows that the minimization problem (10) has a unique solution.

Let us now assume that $\Omega = B_R$ and $f = 1$. A straightforward computation shows that the radial function

$$u(x) = \frac{R^{p'} - |x|^{p'}}{N^{\frac{1}{p-1}} p'} + \frac{m}{\omega_N N^{p'} R^{N-p'}} \tag{11}$$

is the unique solution to the Euler-Lagrange equation associated to (10), which reads

$$\begin{cases} -\Delta_p u = 1 & \text{in } B_R, \\ 0 \in m^{p-1} |Du|^{p-2} \dfrac{\partial u}{\partial \nu} + \phi(u) \left(\displaystyle\int_{\partial B_R} u \, d\mathcal{H}^{n-1} \right)^{p-1} & \text{on } \partial B_R, \end{cases}$$

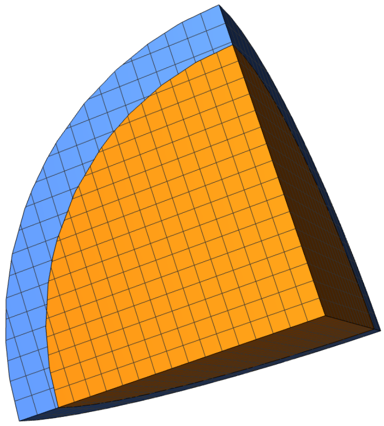

Fig. 3 Representation of B_1 in $\mathbb{R}^3$ with insulator material of constant thickness $h(x) = \frac{1}{4\pi}$.

where $\phi(u)$ is the subdifferential of the real valued function $|t|$, $t = \phi(u)$.

Note that the appearance of the incursion $0 \in \dots$ in the Euler-Lagrange equation is due to the non differentiability of the functional

$$u \mapsto \int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}.$$

We conclude that when $\Omega = B_R$ we have

$$h(x) = m \frac{|u(x)|}{\left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1}\right)} = \frac{m}{N\omega_N R^{N-1}}.$$

□

It is possible to obtain a closed-form expression for

$$E(h) = \min_{u \in W^{1,p}(\Omega)} F_h(u).$$

Specifically, if u is a solution of (5), then by taking u as a test function in the weak formulation (4), one obtains

$$E(h) = \frac{1-p}{p} \int_\Omega fu.$$

In particular, when $f \equiv 1$, minimizing $E(h)$ is equivalent to maximizing the average value of u over Ω. In the linear case, $p = 2$, the solution u is a model for temperature, and in this case, the solution maximizes the average temperature in Ω (see [6] for more).

A natural question is: what is the optimal domain for the minimization problem (5)? That is, if we allow the domain Ω to vary among all Lipschitz domains with fixed volume, which one minimizes (5)? This problem can be seen as a type of isoperimetric inequality and remains open for $p \neq 2$ (it has been resolved in the case $p = 2$; see [14]). The following Theorem answer this question if when $p \neq 2$.

Theorem 3. *Suppose $p \geq 2$, $f \equiv 1$, $m > 0$ fixed, and let (u, h) be the solution pair obtained in Theorem 2. Then*

$$\int_\Omega u\,dx \leq \frac{1}{N^{p'}\omega_N^{\frac{p'}{N}}}\left(\frac{N(p-1)}{p+N(p-1)}|\Omega|^{\frac{p+N(p-1)}{N(p-1)}} + |\Omega|^{\frac{p}{N(p-1)}}m\right),$$

and equality holds if and only if Ω is a ball.

Proof. For $t > 0$, define

$$\begin{aligned} U_t &= \{x \in \Omega; u(x) > t\}, \\ \partial U_t^{\text{int}} &= \partial U_t \cap \Omega, \text{ and } \partial U_t^{\text{ext}} = \partial U_t \cap \partial\Omega, \\ \mu(t) &= |U_t|, \\ P(t) &= \operatorname{Per}(U_t). \end{aligned}$$

Now, given $t, k > 0$, consider the test function φ in (4), defined as

$$\varphi = \begin{cases} 0, & \text{if } 0 < u < t, \\ u - t, & \text{if } t < u < t + k, \\ k, & \text{if } u > t + k, \end{cases}$$

We obtain

$$\begin{aligned} &\int_{U_t \setminus U_{t+k}} |Du|^p\,dx + k\int_{\partial U_{t+h}^{\text{ext}}} \frac{|u|^{p-2}u}{h^{p-1}}\,d\mathcal{H}^{N-1} \\ &+ \int_{\partial U_t^{\text{ext}} \setminus \partial U_{t+h}^{\text{ext}}} \frac{|u|^{p-2}u}{h^{p-1}}(u-t)\,d\mathcal{H}^{N-1} = \int_{U_t \setminus U_{t+k}} (u-t)\,dx + k\int_{U_{t+k}} dx \end{aligned}$$

Dividing both sides by k and letting $k \to 0$ we have

$$\mu(t) = \int_{\partial U_t^{\text{int}}} |Du|^{p-1}\,d\mathcal{H}^{N-1} + \int_{\partial U_t^{\text{ext}}} \frac{|u|^{p-2}u}{h^{p-1}}\,d\mathcal{H}^{N-1}.$$

If we set

$$g(x) = \begin{cases} |Du|^{p-1}, & \text{if } x \in \partial U_t^{\text{int}}, \\ \frac{|u|^{p-2}u}{h^{p-1}}, & \text{if } x \in \partial U_t^{\text{ext}} \end{cases},$$

then the above expression becomes

$$\mu(t)=\int_{\partial U_t} g\, d\mathcal{H}^{N-1}.$$

Applying Hölder's inequality, we obtain

$$\begin{aligned} P(t)^2 &\le \left(\int_{\partial U_t} \sqrt[p-1]{|g|}\, d\mathcal{H}^{N-1}\right)\left(\int_{\partial U_t} \frac{1}{\sqrt[p-1]{|g|}}\, d\mathcal{H}^{N-1}\right) \\ &\le \mu(t)^{\frac{1}{p-1}} P(t)^{\frac{p-2}{p-1}} \left(-\mu'(t)+\int_{\partial U_t^{\text{ext}}} \frac{h}{|u|}\, d\mathcal{H}^{N-1}\right). \end{aligned} \tag{12}$$

Recall the isoperimetric inequality:

$$\left(\frac{\mu(t)}{\omega_N}\right)^{N-1} \le \left(\frac{P(t)}{N\omega_N}\right)^{N}.$$

Combining this with (12), we obtain

$$N^{p'}\omega_N^{\frac{p'}{N}}\mu(t)^{\left(2-\frac{p-2}{p-1}\right)\left(1-\frac{1}{N}\right)-\frac{1}{p-1}} \le -\mu'(t)+\int_{\partial U_t^{\text{ext}}} \frac{h}{|u|}\, d\mathcal{H}^{N-1},$$

or equivalently,

$$\mu(t) \le \frac{1}{N^{p'}\omega_N^{\frac{p'}{N}}}\left(-\mu'(t)\mu(t)^{\frac{p}{N(p-1)}}+\mu(t)^{\frac{p}{N(p-1)}}\int_{\partial U_t^{\text{ext}}} \frac{h}{|u|}\, d\mathcal{H}^{N-1}\right).$$

Finally, integrating from 0 to $+\infty$, we have

$$\int_\Omega u\, dx = \int_0^{+\infty} \mu(t)\, dt \le \frac{1}{N^{p'}\omega_N^{\frac{p'}{N}}}\left(\frac{N(p-1)}{p+N(p-1)}|\Omega|^{\frac{p+N(p-1)}{N(p-1)}}+|\Omega|^{\frac{p}{N(p-1)}}m\right).$$

The explicit solution provided in (11) achieves equality in the inequality above. □

Remark 1. *Although the above proof is not valid for $1<p<2$ due to the lack of well-definedness of certain expressions, we expect that a modified version of the argument can be developed to address this range of p values.*

3 The eigenvalule problem

Theorem 4. *The minimization problem (7) admits a solution.*

Proof. The proof of Theorem 2 can be readily adapted to this setting. Specifically, the problem is equivalent to minimizing the functional

$$J(u) = \frac{\displaystyle\int_\Omega |Du|^p \, dx + \frac{1}{m^{p-1}} \left(\int_{\partial\Omega} |u| \, d\mathcal{H}^{N-1} \right)^p}{\displaystyle\int_\Omega |u|^p \, dx},$$

which admits a minimizer by standard arguments from the Calculus of Variations. On the other hand, since the functional $J(u)$ is not strictly convex, uniqueness of the minimizer is not guaranteed.

□

As discussed above, an interesting open problem is to analyze the minimization problem (7) when the Lipschitz domain Ω is allowed to vary.

Remarkably, the following non-existence result holds for certain values of p in this setting.

Theorem 5. *If $\frac{p+1}{p-1} \leq N$, then the minimization problem (8) does not admit a solution.*

Proof. As before, the problem is equivalent to the minimization problem

$$X = \inf \left\{ \frac{\displaystyle\int_\Omega |Du|^p \, dx + \frac{1}{m^{p-1}} \left(\int_\Omega |u| \right)^p}{\displaystyle\int_\Omega |u|^p \, dx} : u \in W^{1,p}(\Omega) \setminus \{0\}, \ |\Omega| \leq 1 \right\}.$$

Taking $u \equiv 1$ and considering the sequence $\Omega_k = B_{1/k}$, we obtain

$$X \leq \frac{\left(\mathcal{H}^{N-1}(\partial B_{1/k})\right)^p}{m^{p-1}|B_{1/k}|} = \frac{N^p \omega_N^{p-1}}{m^{p-1} k^{(p-1)N-p}}.$$

Letting $k \to +\infty$, it follows that $X = 0$. Hence, the infimum is not attained by any open set $\Omega \subset \mathbb{R}^N$, and the minimization problem (8) does not admit a solution. □

Remark 2. *If $p = 2$, then $\frac{p+1}{p-1} = 3$, and we recover the result known in the linear case. Similarly, if $p \geq 3$, then (8) admits no solution for $N \geq 2$. On the other hand, when*

$$2 \leq N < \frac{p+1}{p-1},$$

we believe that the arguments used in the proof of Theorem 3 can be adapted to establish a Faber–Krahn-type isoperimetric inequality in this regime (see [9] for a related problem). In particular, we conjecture that the ball remains a minimizer in this setting as well. Complete proofs related to this Faber–Krahn-type isoperimetric inequality will be presented elsewhere.

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