A NEW PROOF OF THE $C^{p'}$-CONJECTURE IN THE PLANE VIA A PRIORI ESTIMATES

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Abstract. In this note we discuss an alternative proof that weak solutions to

$$-\Delta_p u = f \in L^\infty(B_1)$$

are in $C^{p'}_{loc}(\Omega)$, where $p > 2$ and $\Omega \subseteq \mathbb{R}^2$. The first complete proof of this fact was given in [1], here we give an alternative argument.

Introduction

The purpose of this work is to prove the optimal regularity of weak solutions to p-Poisson equation

$$-\Delta_p u = f \tag{1}$$

in a bounded domain $\Omega \subseteq \mathbb{R}^2$ (which we will assume is the open unit ball $\Omega = B_1$), with a bounded source $f \in L^\infty(B_1)$, where we assume $p > 2$.

A solution to (1) is given by a function $u \in \mathbf{W}^{1,p}(B_1)$ satisfying:

$$\int_{B_1} |Du|^{p-2} Du D\varphi = \int_{B_1} f\varphi, \quad \forall \varphi \in \mathcal{C}_0^\infty(B_1)$$

Whenever we take $f = 0$ in the definition above, we say u is p-harmonic. The first major result concerning the regularity of p-harmonic functions was given in [5], namely, $u \in C^{1,\alpha}_{loc}$ for some small $\alpha > 0$. In [2], the author gives an alternative and simplified proof of the same result.

It is natural to ask what is the optimal value for the α described above. In the two dimensional case, the authors of [3] found the optimal regularity of p-harmonic functions. The proof given in [3] suggested that p' could have an important hole when f is taken to be nontrivial. Here $p' = 1 + \frac{1}{p-1}$ denotes the Holder conjugate of p.

If f is constant, the trivial example $u(x) = |x|^{p'}$ shows that it's reasonable to expect that $\alpha \leq \frac{1}{p-1}$. This motivates the following:

Conjecture 1. *Let $p > 2$ and $f \in L^\infty(B_1)$. The optimal regularity for a weak solution $u \in \mathbf{W}^{1,p}(B_1)$ to the equation*

$$-\Delta_p u = f$$

is $u \in C^{p'}_{loc}(B_1)$.

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In [4], the conjecture is almost solved, in the sense that it was shown that $u \in C^{p'-\epsilon}_{loc}(B_1)$ for every $\epsilon > 0$. Using different arguments, the conjecture was finally solved in [1]. The idea of the proof was to deform solutions of (1) into p-harmonic functions and to use the known regularity of p-harmonic functions given by [3].

Using a different approach we shall give a proof of the following theorem.

Theorem 1. *Let $p > 2$. Then the p'-conjecture is true in the plane.*

The idea of our method is based on a-priori estimates.

Proof of Theorem 1

It's enough to prove the following:

Lemma A. *Let $p > 2$ and $u(x)$ be a $C^{p'}$ weak solution to the equation*

$$-\Delta_p u = f,$$

where $f \in L^\infty(B_1)$. Then there exists $C > 0$, independent of u, such that

$$\|u\|_{C^{p'}(B_{\frac{1}{2}})} \leq C\left(\|u\|_\infty + \|f\|_\infty^{\frac{1}{p-1}}\right).$$

Proof. By interpolation, It suffices to show that for any delta $\delta > 0$ there exists $C_\delta > 0$ such that

$$[Du]_{C^{\frac{1}{p-1}}(B_{\frac{1}{2}})} \leq \delta [Du]_{C^{\frac{1}{p-1}}(B_1)} + C_\delta\left(\|Du\|_\infty + \|f\|_\infty^{\frac{1}{p-1}}\right) \tag{2}$$

Suppose that 2 doesn't hold. Then there exists $\delta > 0$, such that for each $k \in \mathbb{N}$, we can find u_k and f_k such that

$$-\Delta_p u_k = f_k$$

but

$$[Du_k]_{C^{\frac{1}{p-1}}(B_{\frac{1}{2}})} > \delta [Du_k]_{C^{\frac{1}{p-1}}(B_1)} + k\left(\|Du_k\|_\infty + \|f_k\|_\infty^{\frac{1}{p-1}}\right) \tag{3}$$

Choose $x_k, y_k \in B_{\frac{1}{2}}$ such that

$$\frac{|Du_k(x_k) - Du_k(y_k)|}{|x_k - y_k|^{\frac{1}{p-1}}} \geq \frac{1}{2}[Du_k]_{C^{\frac{1}{p-1}}(B_{\frac{1}{2}})} \tag{4}$$

If we set $r_k := \frac{|x_k - y_k|}{2}$ and $s_k := \frac{x_k + y_k}{2}$, we can use (3) to conclude that $r_k \to 0$.

Consider the second order increment around s_k:

$$\tilde{u_k}(x) = \frac{u_k(s_k + r_k x) + u_k(s_k - r_k x) - 2u_k(s_k)}{r_k^{p'}[Du_k]_{C^{\frac{1}{p-1}}(B_1)}} \tag{5}$$

defined on $B_{\frac{1}{2r_k}}$ We obviously have $\tilde{u_k}(0) = 0$ and $D\tilde{u_k}(0) = 0$. Moreover, if we set $z_k = \frac{x_k - y_k}{2r_k} \in \mathbb{S}^{n-1}$, then

$$|\tilde{u_k}(x)| \leq 2|x|^{p'} \text{ and } [D\tilde{u_k}]_{C^{\frac{1}{p-1}}(B_{\frac{1}{2r_k}})} \leq 2 \tag{6}$$

by (4) we have

$$|D\tilde{u_k}(z_k)| > \frac{\delta}{2} \tag{7}$$

It follows that $\tilde{u_k}$ is locally bounded, and bounded in the $C^{p'}$-norm, hence by Arzela-Ascoli, it converges (up to a subsequence) locally uniformly to a $C^{p'}$-function $\tilde{u}(x)$ defined on the whole $\mathbb{R}^n$.

Let $z_k \to z$ (up to a subsequence), then

$$D\tilde{u}(0) = 0, [D\tilde{u}]_{C^{\frac{1}{p-1}}(\mathbb{R}^n)} \leq 2 \text{ and } |D\tilde{u}(z)| > \frac{\delta}{2}. \tag{8}$$

We claim $\tilde{u}$ is p-harmonic in $\mathbb{R}^n$. Consider $-\Delta_p \tilde{u_k} = \tilde{f_k}$ for $x \in B_{1/2r_k}$. For $\varphi \in \mathcal{C}_0^\infty(B_1)$, take k large enough such that $\operatorname{supp}\varphi \subseteq B_{1/2r_k}$, we have:

(9)

$$\begin{aligned}\int |D\tilde{u_k}|^{p-2} D\tilde{u_k} D\varphi \leq & \int \frac{|Du_k(s_k + r_k x)|^{p-2} + |Du_k(s_k - r_k x)|^{p-2}}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}} \cdot [Du_k(s_k + r_k x) - Du_k(s_k - r_k x)]\, D\varphi \\ =& I + II + III + IV\end{aligned}$$

We analyze I, II, III and IV separately. We have:

$$\begin{aligned}|I| &= \left| \int \frac{|Du_k(s_k + r_k x)|^{p-2} Du_k(s_k + r_k x) D\varphi(x)}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}} \right| \\ &= \left| \int \frac{|Du_k(y)|^{p-2} Du_k(y) D_y \varphi((y - s_k)/r_k) r_k^{1-n}}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}} \right| \\ &\leq \int \frac{|f_k(s_k + r_k x)\varphi(x)|}{[Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}}, \text{ using Holder's inequality we obtain} \\ &\leq \frac{\|f_k\|_\infty \|\varphi\|_1}{[Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}}, \text{ now by (3) we have} \\ &< \frac{\|\varphi\|_1}{k^{p-1}}.\end{aligned} \tag{10}$$

In a complete analogous way, we obtain

$$|IV| \leq \frac{\|\varphi\|_1}{k^{p-1}}.$$

Notice that $|II| = |III|$. Also,

(11)

$$|II| = \left| - \int \frac{|Du_k(s_k + r_k x) - Du_k(s_k - r_k x) + Du_k(s_k - r_k x)|^{p-2} Du_k(s_k - r_k x) D\varphi(x)}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}} \right|$$

$$\leq \int \frac{[r_k |x|]^{\frac{p-2}{p-1}} [Du_k]^{p-2}_{C^{\frac{1}{p-1}}(B_1)} |Du_k(s_k - r_k x) D\varphi(x)|}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}}$$

$$+ \left| \int \frac{|Du_k(s_k - r_k x)|^{p-2} Du_k(s_k - r_k x) D\varphi(x)}{r_k [Du_k]^{p-1}_{C^{\frac{1}{p-1}}(B_1)}} \right|, \text{ using (3) we obtain}$$

$$\leq \left(\frac{1}{2}\right)^{\frac{p-2}{p-1}} \frac{\|D\varphi\|_1}{k} + \frac{\|\varphi\|_1}{k^{p-1}}$$

We conclude that when $k \to \infty$, $\tilde{f}_k \to 0$. Therefore, $\tilde{u}(x)$ satisfies $-\Delta_p \tilde{u}(x) = 0$, i.e. $\tilde{u}$ is p-harmonic.

The following lemma is proved in [4]

Lemma B. ***(Liouville's Theorem)*** *Let $p > 2$. If u is an entire p-harmonic function in $\mathbb{R}^2$ satisfying*

$$\|u\|_\infty \leq C R_j^{p'}$$

for some sequence $R_j \to \infty$ then Du is constant.

By (6) and the lemma above, $D\tilde{u}$ is constant, a contradiction to (7)

□

Let u satisfies $-\Delta_p u = f$ and ρ_δ be the standard mollifier. Consider the boundary problem

$$\begin{cases} \Delta_p u_\epsilon + \epsilon \Delta u_\epsilon = f * \rho_\delta \text{ in } B_1 \\ u_\epsilon = u * \rho_\delta \text{ on } \partial B_1 \end{cases}$$

One can easily modify the argument above, and conclude by lemma A that

$$\|u_\epsilon\|_{C^{p'}(B_{\frac{1}{2}})} \leq C \left(\|u_\epsilon\|_\infty + \|f * \rho_\delta\|_\infty^{\frac{1}{p-1}} \right).$$

On the other hand, elementary properties of mollifiers leads to

$$\|u_\epsilon\|_\infty \leq \|u\|_\infty \text{ and } \|f * \rho_\delta\|_\infty \leq \|f\|_\infty$$

We conclude that $\|u_\epsilon\|_{C^{p'}(B_{\frac{1}{2}})}$ is uniformly bounded for every $\epsilon > 0$, so up to a subsequence, $u_\epsilon \to v$ locally uniformly when $\epsilon, \delta \to 0$, moreover $v \in C^{p'}(B_{\frac{1}{2}})$. By uniqueness of the solution, we must have $u \equiv v$.

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