

Real Analysis: Functions of a real variable

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1 Sets

A **set** X is a collection of objects, also called the *elements* of the set. If ‘ a ’ is an element of X , we write $a \in X$. On the other hand, if ‘ a ’ isn’t an element of X , we write $a \notin X$.

A set X is *well defined* when there is a rule that allows us to say if an arbitrary element ‘ a ’ is or isn’t an element of X .

Example 1. *The set X of all right triangles is well-defined. Indeed, given any object ‘ a ’, if ‘ a ’ is not a triangle or doesn’t have a right angle then $a \notin X$. If ‘ a ’ is a right triangle then $a \in X$.*

Example 2. *The set X of all tall people is not well-defined. The notion of ‘tall’ is not universally defined, hence given any element a we can’t say if $a \in X$ or $a \notin X$.*

Usually one uses the notation

$$X = \{a, b, c, \dots\}$$

to represent the set X whose elements are a, b, c, \dots , and if a set has no elements we denote it by \emptyset and call it the **empty set**.

The set of *natural numbers* $1, 2, 3, \dots$ will be represented by

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

The set of *integers* will be represented by

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

The set of *rational numbers*, that is, fractions $\frac{a}{b}$, where $a, b \in \mathbb{Z}$ and $b \neq 0$, will be denoted by

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

The vast majority of sets in mathematics are not defined by specifying its elements one by one. What usually happens is a set being defined by some property its elements satisfy, i.e. if a has property P then $a \in X$, whereas if a doesn't have property P then $a \notin X$. One writes

$$X = \{a \mid a \text{ has property } P\}$$

For example, the set

$$X = \{a \in \mathbb{N} \mid a > 10\},$$

consists of all natural numbers bigger than 10.

Given two sets A, B , one says that A is a **subset** of B or that A is *included* in B (B *contains* A), represented by $A \subseteq B$, if every element of A is an element of B .

Example 3. *We have the obvious inclusion of sets:*

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q}.$$

Example 4. *Let X be the set of all squares and Y be the set of all rectangles. Then $X \subseteq Y$, since every square is a rectangle.*

When one writes $X \subseteq Y$, it's possible that $X = Y$. In case $X \neq Y$, we say X is a *proper subset*, the notation $X \subsetneq Y$ is sometimes used to indicate that X is a proper subset of Y .

Notice that to write $a \in X$ is equivalent to say $\{a\} \subseteq X$. Also, by definition, it's always true that $\emptyset \subseteq X$ for every set X .

It's easy to see that the inclusion of sets has the following properties:

1. *Reflexive*, $X \subseteq X$ for every set X ;
2. *Anti-symmetric*, if $X \subseteq Y$ and $Y \subseteq X$ then $X = Y$;
3. *Transitive*, if $X \subseteq Y$ and $Y \subseteq Z$ then $X \subseteq Z$.

It follows that two sets X and Y are the same if and only if $X \subseteq Y$ and $Y \subseteq X$, that is to say, they have the same elements.

Given a set X , we define the *power set* of X , $\mathcal{P}(X)$ as

$$\mathcal{P}(X) = \{A \mid A \subseteq X\}.$$

The set $\mathcal{P}(X)$ is the set of all subsets of X , in particular it's never empty, it has at least \emptyset and X itself as elements.

Example 5. Let $X = \{1, 2, 3\}$ then

$$\mathcal{P}(X) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\}.$$

Notice that by using the Fundamental Counting Principle, any set with n elements has 2^n subsets. Therefore, the number of elements of $\mathcal{P}(X)$ is 2^n .

2 Operation with sets

We given two sets X and Y , one can build many other sets. For example, the **union** of X and Y , denoted by $X \cup Y$ is the of elements that are in X or Y , more precisely:

$$X \cup Y = \{a \mid a \in X \text{ or } a \in Y\}.$$

Similarly, the **intersection** of X and Y , denoted by $X \cap Y$ is the of elements that are common to both X and Y :

$$X \cap Y = \{a \mid a \in X \text{ and } a \in Y\}.$$

If $X \cap Y = \emptyset$, then X and Y are said to be *disjoint*.

Example 6. Let $X = \{a \in \mathbb{N} \mid a \leq 100\}$ and $Y = \{a \in \mathbb{N} \mid a > 50\}$ then

$$X \cup Y = \mathbb{N} \text{ and } X \cap Y = \{a \in \mathbb{N} \mid 50 < a \leq 100\}$$

Example 7. The sets $X = \{a \in \mathbb{N} \mid a > 1\}$ and $Y = \{a \in \mathbb{N} \mid a < 2\}$ are *disjoint*, i.e. $X \cap Y = \emptyset$ since there is no natural number between 1 and 2.

The **difference** between X and Y , denoted by $X - Y$ is the set of elements that are in X but not in Y , more precisely:

$$X - Y = \{a \mid a \in X \text{ and } a \notin Y\}.$$

Given an inclusion of sets $X \subseteq Y$, the **complement** of X in Y is the set $Y - X$, the notation X^c sometimes is used if there is no confusion about who the set Y is.

Example 8. Consider the sets $X = \{a \in \mathbb{N} \mid a \text{ is even}\}$ and $Y = \mathbb{N}$. Then $X \subseteq Y$ and $X^c = \{a \in \mathbb{N} \mid a \text{ is odd}\}$.

Proposition 9. *Given sets A, B, C, D the following properties are true:*

1. $A \cup \emptyset = A; A \cap \emptyset = \emptyset$
2. $A \cup A = A; A \cap A = A$
3. $A \cup B = B \cup A; A \cap B = B \cap A$
4. $A \cup (B \cap C) = (A \cup B) \cap C; A \cap (B \cup C) = (A \cap B) \cup C$
5. $A \cup B = A \Leftrightarrow B \subseteq A; A \cap B = A \Leftrightarrow A \subseteq B$
6. *if $A \subseteq B$ and $C \subseteq D$ then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$*
7. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C); A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
8. $(A^c)^c = A$
9. $(A \cup B)^c = A^c \cap B^c; (A \cap B)^c = A^c \cup B^c$

Proof. The last property, $(A \cup B)^c = A^c \cap B^c$, will be demonstrated below, the others are trivial or can be proved in a similar way.

We prove that $(A \cup B)^c \subseteq A^c \cap B^c$. Let $a \in (A \cup B)^c$, then $a \notin A \cup B$, in particular, $a \notin A$ and $a \notin B$, hence $a \in A^c \cap B^c$.

Conversely, take $a \in A^c \cap B^c$. Then $a \notin A$ and $a \notin B$, so $a \notin A \cup B$ and it follows that $a \in (A \cup B)^c$. \square

An *ordered pair* (a, b) is formed by two objects a and b , such that for any other such pair (c, d) :

$$(a, b) = (c, d) \Leftrightarrow a = c \text{ and } b = d.$$

The elements a and b are called *coordinates* of (a, b) , a is the first coordinate and b the second one.

The **cartesian product** $X \times Y$ of two sets X and Y is the set of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$:

$$X \times Y = \{ (x, y) \mid x \in X \text{ and } y \in Y \}.$$

Remark 1. *An ordered pair is not the same as a set, i.e. $(a, b) \neq \{a, b\}$. Notice that $\{a, b\} = \{b, a\}$ but $(a, b) \neq (b, a)$ in general.*

Example 10. *Consider the sets $X = \{1, 2, 3\}$ and $Y = \{a, b\}$, then*

$$X \times Y = \{ (1, a), (1, b), (2, a), (2, b), (3, a), (3, b) \}.$$

3 Functions

A **function** $f : X \rightarrow Y$ consists of three components: a set X , the *domain*, a set Y , the *co-domain*, and a rule that associates each element $a \in X$ an unique element in $f(a) \in Y$, $f(a)$ is called the *value* of $f(x)$ at a , or the image of a under $f(x)$.

Another common notation to denote a function is $x \mapsto f(x)$. In this case the domain and codomain can be identified by the context.

Example 11. *The function $f : \mathbb{N} \rightarrow \mathbb{N}$ given by $f(n) = n + 1$ is called the successor function.*

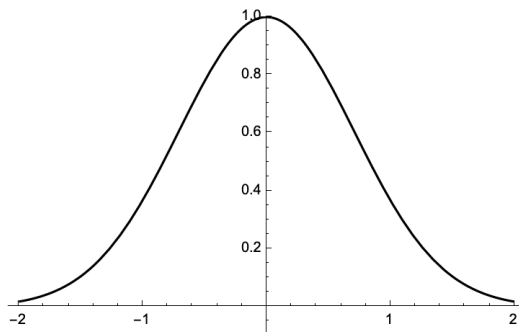
Example 12. *Let X be the set of all triangles. One can define a function $f : X \rightarrow \mathbb{R}$ by $f(x) = \text{area of } x$.*

Example 13. *(Relation that is not a function) The correspondence that associates to each real number x , all y satisfying $y^2 = x$ is not a function because any $x \neq 0$ will be associated to two values, namely $\pm\sqrt{x}$, and in order to be a function every x has to have exactly one image $y = f(x)$.*

The graph of a function $f : X \rightarrow Y$ is a subset of $X \times Y$ defined by

$$\Gamma(f) = \{ (x, f(x)) \mid x \in X \}.$$

Example 14. *Consider the function $f(x) = e^{-x^2}$, its graph is given below:*

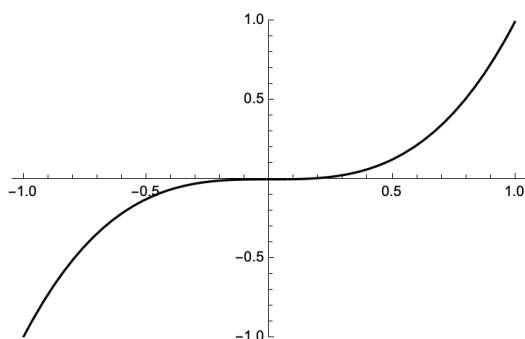


A function $f : X \rightarrow Y$ is said to be *injective* or *one-to-one* if for every x, y such that $f(x) = f(y)$ then $x = y$. Suppose $X \subseteq Y$, then inclusion $i : X \rightarrow Y$ given by $i(x) = x$ is a typical example of injective function.

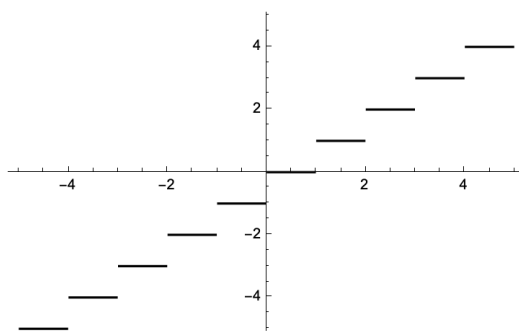
A function $f : X \rightarrow Y$ is said to be *surjective* or *onto* if for every $y \in Y$ there is $x \in X$ such that $y = f(x)$. The projection $p : X \times Y \rightarrow X$ in the first coordinate, given by $p(x, y) = x$ is a typical example of surjection.

Finally, a function $f : X \rightarrow Y$ is *bijective or a bijection* if it is both surjective and injective.

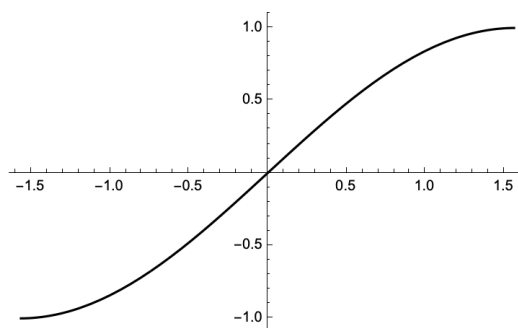
Example 15. The function given by $f(x) = x^3$ is injective.



Example 16. The step function $f(x) = \max\{n \in \mathbb{Z} \mid n \leq x\}$ is not injective.



Example 17. The function $f(x) = \sin x$ is a bijection if we consider $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$.



Given a function $f : X \rightarrow Y$, the *image* of a set $A \subseteq X$ is defined by

$$f(A) = \{y \in Y \mid y = f(a), a \in A\}.$$

Conversely, the *inverse image* of a set (sometimes called *pre-image*) $B \subseteq Y$ is given by

$$f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

Proposition 18. *Given $f : X \rightarrow Y$ and subsets $A, B \subseteq X$, we have:*

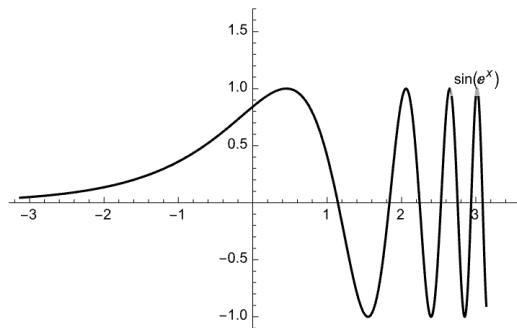
1. $f(A \cup B) = f(A) \cup f(B)$; $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
2. $f(A \cap B) \subseteq f(A) \cap f(B)$; $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
3. if $A \subseteq B$ then $f(A) \subseteq f(B)$ and $f^{-1}(A) \subseteq f^{-1}(B)$
4. $f(\emptyset) = \emptyset$; $f^{-1}(\emptyset) = \emptyset$
5. $f^{-1}(Y) = X$
6. $f^{-1}(A^c) = (f^{-1}(A))^c$

Example 19. *Consider the function $f(x, y) = x^2 + y^2$, the inverse image $f^{-1}(\{1\})$ is a circle of radius 1. Similarly, any line $ax + by = c$ can be seen as $g^{-1}(\{c\})$, where $g(x, y) = ax + by$.*

Given two functions $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, the *composition* $g \circ f$ of g and f is defined as the function:

$$(g \circ f)(x) = g(f(x))$$

Example 20. *The composition of the functions $g(x) = \sin x$ and $f(x) = e^x$ is the function $(g \circ f)(x) = \sin e^x$ depicted below.*



Given a function $f : X \rightarrow Y$ and a subset $A \subseteq X$, the restriction of $f(x)$ to A , denoted by $f|_A : A \rightarrow Y$, is defined by $f|_A(x) = f(x)$. Similarly, if $X \subseteq Z$, a *extension* of $f(x)$ to Z is any function $g : Z \rightarrow Y$ such that $g|_X(x) = f(x)$.

Example 21. Consider again the function $f(x, y) = x^2 + y^2$, and the unit circle $\mathbb{S}^1 = \{(x, y) \mid x^2 + y^2 = 1\}$. Then the restriction $f|_{\mathbb{S}^1}$ is the constant function $g(x) = 1$.

Given functions $f : X \rightarrow Y$, and $g : Y \rightarrow X$, the function $g(x)$ is called *left-inverse* of $f(x)$ if

$$(g \circ f)(x) = x.$$

Similarly, the function $g(x)$ is called *right-inverse* of $f(x)$ if

$$(f \circ g)(x) = x.$$

Finally, if there is a function $f^{-1}(x)$ such that

$$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x,$$

$f^{-1}(x)$ is called the *inverse* of $f(x)$. Notice that any inverse, if exists, is unique. If $g(x)$ and $h(x)$ are both inverses of $f(x)$ then

$$g(x) = g(f(h(x))) = (g \circ f)(h(x)) = h(x).$$

Proposition 22. A function $f : X \rightarrow Y$ has an inverse $f^{-1} : Y \rightarrow X \Leftrightarrow f$ is bijective.

Proof. Suppose f has an inverse f^{-1} and $f(x) = f(y)$ for some x, y . Taking the inverse on both sides, we conclude that $x = y$ and f is injective. Similarly, take $y \in Y$ and set $x = f^{-1}(y)$, then $f(x) = y$ and it follows that f is surjective.

Conversely, suppose f bijective. If $f(x) = y$, set $f^{-1}(y) = x$. One can easily check that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$. \square

Example 23. Consider the function $f : (0, +\infty) \rightarrow (0, +\infty)$ given by $f(x) = \frac{1}{x}$, then the f is its own inverse, i.e. $(f \circ f)(x) = x$.

4 The natural numbers \mathbb{N}

The natural numbers are built axiomatically. Start with a set \mathbb{N} , whose elements are called *natural numbers*, and a function $s : \mathbb{N} \rightarrow \mathbb{N}$, called the *successor function*. For any $n \in \mathbb{N}$, $s(n)$ is called the successor of n .

The function $s(n)$ satisfies the following axioms:

- Axiom 1.** $s(n)$ is injective, i.e. every number has a unique successor.
- Axiom 2.** The set $\mathbb{N} - s(\mathbb{N})$ has only one element, which will be denoted by 1, i.e. every number has a successor and 1 is not a successor of any number.
- Axiom 3.** (Principle of induction) Let $X \subseteq \mathbb{N}$ be a subset with the following property: $1 \in X$ and given $n \in X$, $s(n) \in X$ as well. Then $X = \mathbb{N}$.

Whenever axiom 3 is used to prove a result, the result is said to be proved by induction.

Proposition 24. For any $n \in \mathbb{N}$, $s(n) \neq n$.

Proof. The proof is by induction. Let $X \subseteq \mathbb{N}$ be a subset defined by:

$$X = \{ n \in \mathbb{N} \mid s(n) \neq n \}.$$

By Axiom 2, $1 \in X$. Let $n \in X$, then $s(n) \neq n$. By Axiom 1, $s(s(n)) \neq s(n)$, hence $s(n) \in X$. The proof follows by Axiom 3. □

Given a function $f : X \rightarrow X$, its power f^n is defined inductively. More precisely, if one sets $f^1 = f$ then f^n is defined by:

$$f^{s(n)} = f \circ f^n.$$

In particular, if one sets $2 = s(1), 3 = s(2), \dots$, then $f^2 = f \circ f, f^3 = f \circ f \circ f, \dots$

Now, given two natural numbers $m, n \in \mathbb{N}$, their sum $m+n \in \mathbb{N}$ is defined by:

$$m+n = s^n(m).$$

It follows that $m+1 = s(m)$ and $m+s(n) = s(m+n)$, in particular:

$$m+(n+1) = (m+n)+1$$

More generally, the following can be proved using induction:

Proposition 25. For any $m, n, p \in \mathbb{N}$:

1. (Associativity) $m + (n + p) = (m + n) + p$;
2. (Commutativity) $m + n = n + m$;
3. (Cancellation Law) $m + n = m + p \Rightarrow n = p$;
4. (Trichotomy) Only one of the following can occur: $m = n$, or $\exists q \in \mathbb{N}$ such that $m = n + q$, or $\exists r \in \mathbb{N}$ such that $n = m + r$.

The notion of order among natural numbers can be defined in terms of addition. Namely, one writes

$$m < n,$$

if $\exists q \in \mathbb{N}$ such that $n = m + q$; in the same situation, one also writes $n > m$. Notice in particular that for every $m \in \mathbb{N}$:

$$m < s(m).$$

Finally, one writes $m \geq n$ if $m > n$ or $m = n$ and a similar definition applies to \leq .

Proposition 26. For any $m, n, p \in \mathbb{N}$:

- (I) (Transitivity) $m < n, n < p \Rightarrow m < p$;
- (II) (Trichotomy) Only one of the following can occur: $m = n$, $m < n$ or $m > n$.
- (III) $m < n \Rightarrow m + p < n + p$.

The multiplication operation $m \cdot n$ will be defined in a similar way as $m + n$ was defined. Let $a_m : \mathbb{N} \rightarrow \mathbb{N}$ be the ‘add m ’ function, $a_m(n) = n + m$. Then multiplication of two natural numbers $m \cdot n$ is defined as:

$$\begin{aligned} m \cdot 1 &:= m, \\ m \cdot (n + 1) &:= (a_m)^n(m). \end{aligned}$$

So $m \cdot 2 = a_m(m) = m + m$, $m \cdot 3 = (a_m)^2(m) = m + m + m, \dots$, and it follows that:

$$m \cdot (n + 1) := m \cdot n + m.$$

More generally, the following is true:

Proposition 27. For any $m, n, p \in \mathbb{N}$:

- a. $m \cdot (n \cdot p) = (m \cdot n) \cdot p$;
- b. $m \cdot n = n \cdot m$;
- c. $m \cdot n = p \cdot n \Rightarrow m = p$;
- d. $m \cdot (n + p) := m \cdot n + m \cdot p$;
- e. $m < n \Rightarrow m \cdot p < n \cdot p$.

5 Well-ordering principle

Let $X \subseteq \mathbb{N}$. A number $m \in X$ is called **the minimum element** of X , denoted $m = \min X$, if $m \leq n$ for every $n \in X$. For example, 1 is the minimum of \mathbb{N} ; 100 is the minimum of $\{100, 1000, 10000\}$.

Lemma 28. If $m = \min X$ and $n = \min X$ then $m = n$.

Proof. Since $m \leq p$ for every $p \in X$, $m \leq n$ in particular. Similarly, $n \leq m$ and hence $m = n$. \square

The maximum element is defined similarly: $m = \max X$ if $m \geq n$, $\forall n \in X$. Notice that not all subsets $X \subseteq \mathbb{N}$ have a maximum. In fact, \mathbb{N} itself doesn't have a maximum, since $m < m + 1$ for every $m \in \mathbb{N}$. The lemma above remains valid if we exchange 'minimum' by 'maximum'.

Despite not all subsets of \mathbb{N} having a maximum, they do have a minimum if they are non-empty.

Theorem 29. (*Well-ordering principle*) Let $X \subseteq \mathbb{N}$ be non-empty. Then X has a minimum.

Proof. If $1 \in X$ then 1 is the minimum, so suppose $1 \notin X$. Let

$$I_n = \{m \in \mathbb{N} \mid 1 \leq m \leq n\},$$

and consider the set

$$L = \{n \in \mathbb{N} \mid I_n \subseteq X^c\}.$$

Since $1 \notin X \Rightarrow 1 \in L$. If $n \in L \Rightarrow n + 1 \in L$ then induction would imply $L = \mathbb{N}$, but $L \neq \mathbb{N}$ since $L \subseteq X^c = \mathbb{N} - X$, and $X \neq \emptyset$. We conclude that there is a m_0 such that $m_0 \in L$ but $m_0 + 1 \notin L$. It follows that $m_0 + 1$ is the minimum element of X . \square

Corollary 30. (Strong induction) Let $X \subseteq \mathbb{N}$ be a set with the following property:

$$\forall n \in \mathbb{N}, \text{ if } X \text{ contains all } m < n \Rightarrow n \in X.$$

Then $X = \mathbb{N}$.

Proof. Set $Y = X^c$, the claim is that $Y = \emptyset$. Suppose not, that is, $Y \neq \emptyset$. By the theorem above, Y has a minimum element, say $p \in Y$. But then by hypothesis $p \in X$, a contradiction. \square

Example 31. Strong induction can be used to prove the **Fundamental theorem of Arithmetic**, which says that every number greater than 1 can be written as a product of primes (a number p is **prime** if $p \neq m \cdot n$, with $m < p$ and $n < p$). Indeed, Let $X = \{m \in \mathbb{N} \mid m \text{ is a product of primes}\}$ and $n \in \mathbb{N}$ a given number. If X contains all numbers m such that $m < n$, then if n is prime, $n \in X$; if n is not a prime then $n = p \cdot q$ with $p < n, q < n$, again it follows that $n \in X$. Therefore, strong induction implies $X = \mathbb{N}$.

Let X be any set. A common way of defining a function $f : \mathbb{N} \rightarrow X$ is **by recurrence** (sometimes ‘by induction’ is used), namely, $f(1)$ is given and also a rule that allows one to obtain $f(m)$ knowing $f(n)$ for all $n < m$. Technically, more than one function f could exist satisfying these conditions, however it is known that such a function is unique, the proof of this fact is left as an exercise.

Example 32. (Factorial) The factorial function $f : n \mapsto n!$ can be defined using induction. Set $f(1) = 1$ and $f(n + 1) = (n + 1) \cdot f(n)$. Then $f(2) = 2 \cdot 1$, $f(3) = 3 \cdot 2 \cdot 1$, \dots , $f(n) = n!$.

Example 33. (Arbitrary sums/products) So far the definition of $m + n$ was used, what about $m + n + p$ or $m_1 + \dots + m_n$? In order to define arbitrary sums (or products), one can use induction. Namely,

$$m_1 + \dots + m_n = (m_1 + \dots + m_{n-1}) + m_n;$$

and similarly, for products:

$$m_1 \cdot \dots \cdot m_n = (m_1 \cdot \dots \cdot m_{n-1}) \cdot m_n.$$

6 Finite and Infinite sets

Throughout this section, I_n stands for the set of numbers less than or equal to n :

$$I_n = \{ m \in \mathbb{N} \mid 1 \leq m \leq n \}$$

A arbitrary set X is **finite** if $X = \emptyset$ or there is number $n \in \mathbb{N}$ and a bijection

$$f : I_n \rightarrow X.$$

In the latter case, one says that X has n elements and writes:

$$|X| = n,$$

f is said to be a counting function for X . By convention, if $X = \emptyset$ then one says X has zero elements, i.e. $|\emptyset| = 0$.

It remains to show that the number of elements is a well-defined notion, that is to say, if there are bijections $f : I_n \rightarrow X$ and $g : I_m \rightarrow X$ then $n = m$.

Theorem 34. *Let $X \subseteq I_n$. If there is a bijection $f : I_n \rightarrow X$, then $X = I_n$.*

Proof. The proof is by induction on n . The case $n = 1$ is obvious, suppose the result true for n , the proof follows if one can prove the result for $n + 1$.

Suppose $X \subseteq I_{n+1}$ and there is a bijection $f : I_{n+1} \rightarrow X$. Let $a = f(n+1)$ and consider the restriction $f : I_n \rightarrow X - \{a\}$.

If $X - \{a\} \subseteq I_n$ then $X - \{a\} = I_n$, $a = n + 1$ and $X = I_{n+1}$.

Suppose $X - \{a\} \not\subseteq I_n$, then $n + 1 \in X - \{a\}$ and one can find b such that $f(b) = n + 1$. Let $g : I_{n+1} \rightarrow X$ be the defined by $g(m) = f(m)$ if $m \neq n + 1, a$; $g(n + 1) = n + 1$; $g(b) = a$. By construction, the restriction $g : I_n \rightarrow X - \{n + 1\}$ is a bijection and obviously $X - \{n + 1\} \subseteq I_n$, hence $X - \{n + 1\} = I_n$ and it follows that $X = I_{n+1}$. \square

Corollary 35. *(Number of elements is well-defined) If there is a bijection $f : I_n \rightarrow I_m$ then $m = n$. Therefore, if $f : I_n \rightarrow X$ and $g : I_m \rightarrow X$ are bijections then $n = m$.*

Proof. The first part follows directly from the theorem. For the second part, consider the composition $(f^{-1} \circ g) : I_m \rightarrow I_n$. \square

Corollary 36. *There is no bijection $f : X \rightarrow Y$ between a finite set X and a proper subset $Y \subseteq X$.*

Proof. By definition there is a bijection $\varphi : I_n \rightarrow X$ for some $n \in \mathbb{N}$. Since Y is proper, $A := \varphi^{-1}(Y)$ is also proper in I_n . Let $\varphi_A : A \rightarrow Y$ be the restriction of φ from I_n to A . Suppose there is a bijection $f : X \rightarrow Y$, then the composite function $\varphi_A^{-1} \circ f \circ \varphi : I_n \rightarrow A$ defines a bijection, a contradiction. \square

Theorem 37. *Let X be a finite set and $Y \subseteq X$, then Y is finite and $|Y| \leq |X|$, the equality occurs only if $X = Y$.*

Proof. It's enough to prove the result for $X = I_n$. If $n = 1$ the result is obvious. Suppose the result is valid for I_n and consider $Y \subseteq I_{n+1}$. If $Y \subseteq I_n$, the induction hypothesis gives the result, so assume $n+1 \in Y$. Then $Y - \{n+1\} \subseteq I_n$ and by induction, there is a bijection $f : I_p \rightarrow Y - \{n+1\}$, where $p \leq n$. Let $g : I_{p+1} \rightarrow Y$ be a bijection defined by $g(n) = f(n)$ if $n \in I_n$, and $g(p+1) = n+1$. This proves that Y is finite, moreover since $p \leq n \Rightarrow p+1 \leq n+1$, $|Y| \leq n$. The last statement says that if $Y \subseteq I_n$ and $|Y| = n$ then $Y = I_n$, but this is a direct consequence of theorem 34. \square

The following Corollary is immediate:

Corollary 38. *Let Y be finite and $f : X \rightarrow Y$ be an injective function. Then X is also finite and $|X| \leq |Y|$.*

Corollary 39. *Let X be finite and $f : X \rightarrow Y$ be an surjective function. Then Y is also finite and $|Y| \leq |X|$.*

Proof. Since f is surjective, by the proof of proposition 22, f has an injective right-inverse $g : Y \rightarrow X$. The result follows by the corollary above. \square

A set X that is not finite is said to be **infinite**. More, precisely X is infinite when it's not empty and there is no bijection $f : I_n \rightarrow X$ for any $n \in \mathbb{N}$.

Example 40. *The natural numbers \mathbb{N} is an infinite set since there is no surjection between I_n and \mathbb{N} , because given any function $f : I_n \rightarrow \mathbb{N}$, the number $f(1) + f(2) + \dots + f(n)$ is not in the range.*

Example 41. *\mathbb{Z} and \mathbb{Q} are also infinite sets since they contain \mathbb{N} , which is infinite.*

A set $X \subseteq \mathbb{N}$ is **bounded**, if there is a number $M \in \mathbb{N}$ such that $n \leq M$ for all $n \in X$.

Theorem 42. *Let $X \subseteq \mathbb{N}$ be nonempty. The following are equivalent:*

- a. X is finite;*
- b. X is bounded;*
- c. X has a greatest element.*

Proof. The proof is based on the implications $a \Rightarrow b$, $b \Rightarrow c$, $c \Rightarrow a$.

($a \Rightarrow b$) Let $X = \{x_1, x_2, \dots, x_n\}$. Then $M = x_1 + \dots + x_n$ satisfies $n \leq M$ for all $n \in X$.

($b \Rightarrow c$) Consider the set $A = \{n \in \mathbb{N} \mid n \geq x, \forall x \in X\}$. Since X is bounded, $A \neq \emptyset$. By the principle of well ordering, A has a minimum element, say $m \in A$. If $m \in X$ then m is the greatest element, so suppose $m \notin X$. By definition, $m > n$ for all $n \in X$, and since $X \neq \emptyset$, $m > 1$, that is $m = p + 1$, for some $p \in \mathbb{N}$. If $p \geq x$ for all $x \in X$ then $p \in A$, a contradiction since $p < m$ and m is minimal. If there is a $x \in X$ such that $x > p$, then $x \geq m$ a contradiction unless $x = m$, but $m \notin X$ by assumption. It follows that $m \in X$ and m is the greatest element.

($c \Rightarrow a$) If X has a greatest element, say M , then $X \subseteq I_M$ and it follows that X is finite.

□

The Theorem below follows directly from the definitions, the proof will be omitted.

Theorem 43. *Let X and Y be two sets such that $|X| = m$, $|Y| = n$ and $X \cap Y = \emptyset$. Then $X \cup Y$ is finite and $|X \cup Y| = m + n$.*

The following corollary is immediate:

Corollary 44. *Let X_1, X_2, \dots, X_n , be a finite collection of sets such that each X_i is finite and $X_i \cap X_j = \emptyset$ if $i \neq j$. Then $\bigcup_{i=1}^n X_i$ is finite and*

$$\left| \bigcup_{i=1}^n X_i \right| = \sum_{i=1}^n |X_i|$$

Corollary 45. Let X_1, X_2, \dots, X_n , be a finite collection of sets such that each X_i is finite. Then $\bigcup_{i=1}^n X_i$ is finite and

$$\left| \bigcup_{i=1}^n X_i \right| \leq \sum_{i=1}^n |X_i|$$

.

Proof. For each $i = 1, \dots, n$, set $Y_i = X_i \times \{i\}$. Then the projection

$$\pi_i : \bigcup_{i=1}^n Y_i \rightarrow \bigcup_{i=1}^n X_i$$

in the first coordinate is surjective, by Corollaries 39 and 44, the proof is complete. \square

Corollary 46. Let X_1, X_2, \dots, X_n , be a finite collection of sets such that each X_i is finite. Then $X_1 \times \dots \times X_n$ is finite and

$$|X_1 \times \dots \times X_n| = \prod_{i=1}^n |X_i|$$

.

Proof. It's enough to prove for $n = 2$, since the general case follows from this one. Let $X_2 = \{y_1, \dots, y_m\}$, notice that $X_1 \times X_2 = X_1 \times \{y_1\} \cup \dots \cup X_1 \times \{y_m\}$, the result follows by Corollary 44. \square

7 Countable Sets

A set X is **countable** if it is finite or there is a bijection $f : \mathbb{N} \rightarrow X$. In the latter case, it is necessarily an infinite set, since as \mathbb{N} is infinite, and we use the term **countably infinite**.

Example 47. The set $X = \{2n \in \mathbb{N} \mid n \in \mathbb{N}\}$ of all even numbers is countable. The function $f(x) = 2x$ defines a bijection between X and \mathbb{N} .

Theorem 48. Let X be an infinite set. Then X has a countably infinite subset.

Proof. It's enough to find an injective function $f : \mathbb{N} \rightarrow X$, since every injective function is a bijection over its image. Choose an element $a_1 \in X$, set $X_1 = X - \{a_1\}$ and $f(1) = a_1$. Since X is infinite, X_1 is also infinite, choose an element a_2 in X_1 , and set $f(2) = a_2$. Proceeding by induction, we have $f(n) = a_n$, $a_n \in X_{n-1}$, where $X_{n-1} = X - \{a_1, a_2, \dots, a_{n-1}\}$.

Suppose $f(n) = f(m)$, with $n, m \in \mathbb{N}$, then $a_n = a_m$, which is possible only if $n = m$. Therefore, f is injective. \square

Corollary 49. *A set X is infinite \iff there is a bijection $f : X \rightarrow Y$, where $Y \subsetneq X$ is a proper subset.*

Proof. (\Rightarrow) Suppose X infinite, by theorem 48, X has a countably infinite subset, say $Z = \{a_1, a_2, a_3, \dots\}$. Set $Y = (X - Z) \cup \{a_2, a_4, a_6, \dots\}$ and define $f(x) = x$ if $x \in X - Z$, and $f(a_n) = a_{2n}$ otherwise. The function $f(x)$, defined this way, is clearly a bijection.

(\Leftarrow) Follows from Corollary 36. \square

A function $f : X \rightarrow Y$ is called *increasing* if $x < y \Rightarrow f(x) < f(y)$.

Theorem 50. *Every subset X of \mathbb{N} is countable.*

Proof. The proof is very similar to the one in theorem 48. If X is finite then is countable, so assume X infinite. We define an increasing bijection $f : \mathbb{N} \rightarrow X$ by induction. Let $X_1 = X$, $a_1 = \min X$ (which exists by Theorem 29), and set $f(1) = a_1$. Now, define $X_2 = X - \{a_1\}$ and $f(2) = a_2 = \min X_2$. By induction, we define $f(n) = a_n = \min X_n$, where $X_n = X - \{a_1, a_2, \dots, a_{n-1}\}$. The function $f(n)$ is injective by construction, suppose $f(n)$ not surjective. There is $x \in X$ such that $x \notin f(\mathbb{N})$. So $x \in X_n$ for every n , which implies that $x > f(n)$ for every n , and x is a bound for the infinite set $f(\mathbb{N})$, a contradiction by Theorem 42. \square

Corollary 51. *Let X be a countable set. Then for any $Y \subseteq X$, Y is countable.*

Corollary 52. *The set of all prime numbers is countable.*

Corollary 53. *Let Y be a countable set and $f : X \rightarrow Y$ an injective function. Then X is countable.*

Corollary 54. *The set \mathbb{Z} of integers is countable.*

Proof. The function $f : \mathbb{Z} \rightarrow \mathbb{N}$ defined by $f(0) = 1, f(m) = 2m, \text{ if } m > 0$ and $f(m) = -2m + 1, \text{ if } m < 0,$ is bijective. \square

Corollary 55. *Let X be a countable set and $f : X \rightarrow Y$ a surjective function. Then Y is countable.*

Proposition 56. *The set $\mathbb{N} \times \mathbb{N}$ is countable.*

Proof. The function defined by $f(m, n) = 2^m 3^n$ is a bijection $f : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}.$ \square

Corollary 57. *Let X_1, X_2, \dots be a countable collection of countable sets. Set $X = \bigcup_{i=1}^{\infty} X_i,$ then X is countable.*

Proof. Let $f_i : \mathbb{N} \rightarrow X_i$ be a counting function for each $i \in \mathbb{N}.$ Then $f(i, m) := f_i(m)$ defines a surjection $f : \mathbb{N} \times \mathbb{N} \rightarrow X.$ By Corollary 55, X is countable. \square

Corollary 58. *If X, Y are countable sets then $X \times Y$ is countable.*

Proof. Let $f_1 : \mathbb{N} \rightarrow X, f_2 : \mathbb{N} \rightarrow Y$ be counting functions. Then $f(m, n) := (f_1(m), f_2(n))$ defines a bijection, Proposition 56 concludes the proof. \square

Corollary 59. *The set \mathbb{Q} of rational numbers is countable.*

Proof. Let \mathbb{Z}^* denote the set of nonzero integers. Define the surjective function $f : \mathbb{Z} \times \mathbb{Z}^* \rightarrow \mathbb{Q}$ given by $f(m, n) = \frac{m}{n}.$ By Corollary 55, \mathbb{Q} is countable. \square

8 Uncountable sets

A set X is **uncountable** if it's not countable. Given two sets X and $Y,$ if there is a bijection $f : X \rightarrow Y,$ we say X and Y have the same **cardinality**, in symbols:

$$\text{card}(X) = \text{card}(Y).$$

If we assume f injective only and there is no surjective function $g : X \rightarrow Y,$ then we say

$$\text{card}(X) < \text{card}(Y).$$

The cardinality of the Natural numbers \mathbb{N} is denoted by

$$\text{card}(\mathbb{N}) = \aleph_0.$$

If the set X is finite with n elements, we say $\text{card}(X) = n$. By definition, for any infinite set X :

$$\aleph_0 \leq \text{card}(X).$$

Recall that given two sets X and Y , the set $\mathcal{F}(X, Y)$ denotes the set of all functions between X and Y .

Theorem 60. (Cantor) *Let X and Y be sets such that Y has at least two elements. There is no surjective function $\phi : X \rightarrow \mathcal{F}(X, Y)$.*

Proof. Suppose a function $\phi : X \rightarrow \mathcal{F}(X, Y)$ is given and let $\phi_x = \phi(x) : X \rightarrow Y$ be the image of $x \in X$, which itself is a function. We claim that there is a $f : X \rightarrow Y$ that is not ϕ_x for any X . Indeed, for each $x \in X$ let $f(x)$ be an element different than $\phi_x(x)$ (this is possible since $|Y| \geq 2$), then $f \neq \phi_x$ for every $x \in X$ and hence, ϕ is not surjective. \square

Corollary 61. *Let X_1, X_2, \dots be a countable collection of countably infinite sets. Then the infinite cartesian product $X = \prod_{i=1}^{\infty} X_i$ is uncountable.*

Proof. It's enough to prove the result for $X_i = \mathbb{N}$. In this case, $X = \mathcal{F}(\mathbb{N}, \mathbb{N})$ and the result follows from Theorem 60. \square

Example 62. *The set $X = \{(a_1, a_2, a_3, a_4, \dots)\}$ of all sequence of natural numbers is uncountable.*

Example 63. *The set of all real numbers \mathbb{R} is uncountable. This will be proved in the next sections.*

9 Fields

A **field** K is a set K together with two operations:

$$+ : K \times K \rightarrow K \text{ and } \cdot : K \times K \rightarrow K$$

satisfying the following properties (also called *field axioms*):

Given $x, y, z \in K$, we have:

1. $(x + y) + z = x + (y + z)$;
2. $x + y = y + x$;
3. There is an element $0 \in K$ such that $\forall x \in K, x + 0 = x$;
4. For any $x \in K$ there is an element $y \in K$ such that $x + y = 0$. We define $-x := y$, and write $z - x$ instead of $z + (-x)$;
5. $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
6. $x \cdot y = y \cdot x$;
7. There is an element $1 \in K$ such that $1 \neq 0$ and $\forall x \in K, x \cdot 1 = x$;
8. For any $x \neq 0$ there is an element $y \in K$ such that $x \cdot y = 1$. We define $x^{-1} := y$, and write $\frac{z}{x}$ instead of $z \cdot x^{-1}$;
9. $x \cdot (y + z) = x \cdot y + x \cdot z$.

Given two fields K and L , we say a function $f : K \rightarrow L$ is a *homomorphism*, if $f(x+y) = f(x)+f(y)$ and $f(c \cdot x) = c \cdot f(x)$. We say f is an *isomorphism* if, in addition, f is bijective and f^{-1} is also a homomorphism. An *automorphism* $f : K \rightarrow K$ is an isomorphism between K and itself.

Example 64. *The set rational numbers \mathbb{Q} together with the operations*

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{db} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

is a field. In this case, $0 = \frac{0}{1}$, $1 = \frac{1}{1}$ and $(\frac{a}{b})^{-1} = \frac{b}{a}$.

Example 65. *If p is prime, the set of integers mod p , $\mathbb{Z}_p = \{\bar{0}, \dots, \overline{p-1}\}$, with operations $\bar{a} + \bar{b} = \overline{a+b}$ and $\bar{a} \cdot \bar{b} = \overline{a \cdot b}$, is a field. It easy to see that $0 = \bar{0}, 1 = \bar{1}$ in this case. Moreover, by Fermat's little theorem $\bar{a} \cdot \bar{a}^{p-2} = \bar{1}$, hence $\bar{a}^{-1} = \bar{a}^{p-2}$.*

Example 66. *The set of rational functions, $\mathbb{Q}(t) = \{ \frac{p(t)}{q(t)} ; p(t), q(t) \in \mathbb{Q}[t], q(t) \neq 0 \}$, where $\mathbb{Q}[t]$ is the set of polynomials with rational coefficients, with the usual operations of fractions is a field.*

Proposition 67. *Let K be a field and $x, y \in K$, then*

- a. $x \cdot 0 = 0$;
- b. $x \cdot z = y \cdot z$ and $z \neq 0$ then $x = y$;
- c. $x \cdot y = 0 \Rightarrow x = 0$ or $y = 0$;
- d. $x^2 = y^2 \Rightarrow x = \pm y$.

Proof. a. Indeed, $x \cdot 0 + x = x \cdot (0 + 1) = x$, hence $x \cdot 0 = 0$.

b. We have $x = x \cdot z \cdot z^{-1} = y \cdot z \cdot z^{-1} = y$.

c. If $x \neq 0$ then $x \cdot y = 0 \cdot x \Rightarrow y = 0$.

d. Notice that $x^2 = y^2 \Rightarrow x^2 - y^2 = 0 \Rightarrow (x - y)(x + y) = 0$.

□

10 Ordered Fields

An ordered field is a field K together with a subset $P \subseteq K$, called the set of *positive elements*, such that for any $x, y \in P$ the following properties hold:

- (I) (*Close under addition/multiplication*) $x + y \in P, x \cdot y \in P$;
- (II) (*Trichotomy*) For any $x \in K$, only one of the following occurs: $x = 0$, $x \in P, -x \in P$.

If we denote $-P = \{-p; p \in P\}$, then K can be written as a disjoint union

$$K = P \cup -P \cup \{0\}$$

Notice that in an ordered field if $x \neq 0$ then $x^2 \in P$. In particular $1 \in P$ in an ordered field.

Example 68. *The field of rational numbers \mathbb{Q} together with the set*

$$P = \left\{ \frac{a}{b} \in \mathbb{Q}; a \cdot b \in \mathbb{N} \right\}$$

is an ordered field.

Example 69. *The field \mathbb{Z}_p can't be ordered, since if we add $\bar{1}$, p times, the result is $\bar{0}$, i.e. $\bar{1} + \dots + \bar{1} = \bar{0}$, but in an ordered field the sum of positive elements has to be positive, in particular nonzero.*

Example 70. The field $\mathbb{Q}(t)$ of example 66 together with the set

$$P = \left\{ \frac{p(t)}{q(t)}; \text{ the leading coefficient of } p(t) \cdot q(t) \text{ is positive} \right\}$$

is an ordered field.

In an ordered field K , if $x - y \in P$ we write $x > y$ (or $y < x$). In particular, $x > 0$ implies $x \in P$ and $x < 0$ implies $x \in -P$. Notice that if $x \in P$ and $y \in -P$ then $x > y$.

We use the notation $x \leq y$ to indicate $x < y$ or $x = y$, in a similar way we can define $x \geq y$ as well.

Proposition 71. Let K be an ordered field and $x, y, z \in K$, then

- (I) (Transitivity) $x < y$ and $y < z \Rightarrow x < z$;
- (II) (Trichotomy) Only one of the following occurs: $x = y$, $x > y$, $x < y$;
- (III) (Sum monotoneity) $x < y \Rightarrow x + z < y + z$;
- (IV) (Multiplication monotoneity) If $z > 0$, then $x < y \Rightarrow x \cdot z < y \cdot z$ and if $z < 0$, then $x < y \Rightarrow x \cdot z > y \cdot z$.

Since in an ordered field K , 1 is always positive we have $1 + 1 > 1 > 0$ and $1 + 1 + 1 > 1 + 1$, so we can easily define an increasing injection

$$f : \mathbb{N} \rightarrow K$$

by $f(n) = \overbrace{1 + 1 + \cdots + 1}^n$, or more precisely, $f(1) = 1$ and $f(n+1) = f(n) + 1$. Therefore, it makes sense to identify \mathbb{N} with $f(\mathbb{N}) \subseteq K$, so henceforward we will simply write

$$\mathbb{N} \subseteq K$$

whenever K is an ordered field.

Notice in particular that $f(n)$ is never zero in this case, hence every ordered field is infinite. Whenever $f(n)$ is never zero, for f defined above, we say K has **characteristic zero**; if $f(p) = 0$, then we say K has **characteristic p**.

Example 72. The field \mathbb{Q} clearly has characteristic zero. The field \mathbb{Z}_p has characteristic p .

Proceeding as before, we can extend the bijection above to $f : \mathbb{Z} \rightarrow K$ and view $\mathbb{Z} \subseteq K$ as well. Hence, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq K$.

Finally, we can use $f : \mathbb{Z} \rightarrow K$ to define a bijection $g : \mathbb{Q} \rightarrow K$ by $g(\frac{a}{b}) = f(a) \cdot f(b)^{-1}$. So we may identify \mathbb{Q} with $g(\mathbb{Q}) \subseteq K$ and write

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq K$$

whenever K is an ordered field.

Example 73. *If $K = \mathbb{Q}$ in the above discussion, then $g : \mathbb{Q} \rightarrow \mathbb{Q}$ is the identity automorphism. i.e. $g(\frac{a}{b}) = \frac{a}{b}$.*

Proposition 74. *(Bernoulli's inequality) Let K be an ordered field and $x \in K$. If $x \geq -1$ and $n \in \mathbb{N}$, then*

$$(1 + x)^n \geq 1 + n \cdot x$$

Proof. We use induction on $n \in \mathbb{N}$. The case $n = 1$ is clear, suppose the result valid for n . Then $(1 + x)^{n+1} = (1 + x)^n(1 + x) \geq (1 + n \cdot x)(1 + x) = 1 + x + n \cdot x + x^2 \geq 1 + x + n \cdot x$, as expected. (Notice that we used the fact that $x \geq -1$ in the first inequality and proposition 71(IV).) \square

11 Intervals

Let K be an ordered field and $a < b$ be elements of K . We call any subset of the following form an interval:

$$[a, b] = \{x \in K; a \leq x \leq b\} \text{ (closed interval)}$$

$$(a, b) = \{x \in K; a < x < b\} \text{ (open interval)}$$

$$[a, b) = \{x \in K; a \leq x < b\} \text{ and } (a, b] = \{x \in K; a < x \leq b\}$$

$$(-\infty, b) = \{x \in K; x < b\} \text{ and } (-\infty, b] = \{x \in K; x \leq b\}$$

$$(a, \infty) = \{x \in K; a < x\} \text{ and } [a, \infty) = \{x \in K; a \leq x\}$$

$$(-\infty, \infty) = K$$

If $a = b$, then $[a, a] = a$ and $(a, a) = \emptyset$. We say the interval $[a, a]$ is degenerate.

Let K be an ordered field and $x \in K$. We define the absolute value of x , denoted by $|x|$, by

$$|x| := \max\{x, -x\},$$

which is to say, $|x|$ is the greater of the two numbers x or $-x$. Geometrically, if the elements of K are put in a straight line, $|x|$ measures the distance between x and 0, hence $|x - a|$ is the distance between x and a .

Theorem 75. *Let x, y be elements of an ordered field K . The following are equivalent:*

- (i) $-y \leq x \leq y$
- (ii) $x \leq y$ and $-x \leq y$
- (iii) $|x| \leq y$

Corollary 76. *Let $x, a, \epsilon \in K$ then*

$$|x - a| \leq \epsilon \iff a - \epsilon \leq x \leq a + \epsilon.$$

Remark 2. *The theorem and corollary remains valid if we exchange \leq by $<$.*

Theorem 77. *Let x, y, z be elements of an ordered field K .*

- (i) $|x + y| \leq |x| + |y|$;
- (ii) $|x \cdot y| = |x| \cdot |y|$;
- (iii) $|x| - |y| \leq ||x| - |y|| \leq |x - y|$;
- (iv) $|x - z| \leq |x - y| + |y - z|$.

Let K be an ordered field and $X \subseteq K$. An **upper bound** of X is an element $M \in K$ such that $x \leq M$ for every $x \in X$. Similarly, a **lower bound** is an element $m \in K$ such that $m \leq x$ for every $x \in X$. We say X is *bounded from above* if it has an upper bound, *bounded from below* if it has a lower bound, and *bounded* if it has upper and lower bounds, i.e. $X \subseteq [m, M]$.

Example 78. *The principle of well-ordering guarantees that \mathbb{N} is bounded from below when viewed as a set inside the ordered field \mathbb{Q} . \mathbb{N} is obviously not bounded from above in \mathbb{Q} , since given any n , $n + 1 > n$.*

Example 79. *Oddly enough, \mathbb{N} is bounded from above in the ordered field $\mathbb{Q}(t)$ from example 70. Since given any $n \in \mathbb{N}$, the rational function $r(t) = t$ satisfies $r(t) - n > 0$. Therefore, $r(t) \in \mathbb{Q}(t)$ is an upper bound for \mathbb{N} and the latter is bounded from above, hence bounded, in $\mathbb{Q}(t)$.*

Theorem 80. *Let K be an ordered field. The following are equivalent:*

1. \mathbb{N} is not bounded from above;
2. Given $a, b \in K$, with $a > 0$, $\exists n \in \mathbb{N}$ such that $n \cdot a > b$;
3. Given $a > 0$ in K , $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < a$.

*A field K satisfying the above conditions is called **Archimedean field**.*

Proof. The proof is based on the implications $1 \Rightarrow 2$, $2 \Rightarrow 3$, $3 \Rightarrow 1$.

(1 \Rightarrow 2) Since \mathbb{N} is unbounded, $\frac{b}{a} < n$ for some $n \in \mathbb{N}$, hence $n \cdot a > b$.

(2 \Rightarrow 3) Take $b = 1$ in 2.

(3 \Rightarrow 1) For any $a > 0$, consider $\frac{1}{a}$, by 3., $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{1}{a} \iff n > a$. Therefore, no positive element is an upper bound. Similarly, no negative element can be an upper bound since if x is negative $-x$ is positive and we can apply the same argument.

□

Example 81. *Examples 78 and 79 say that \mathbb{Q} is Archimedean but $\mathbb{Q}(t)$ isn't.*

12 The real numbers \mathbb{R}

Let K be an ordered field and $X \subseteq K$ be a bounded from above subset. The **supremum** of X , denoted $\sup X$ is the least upper bound of X , in other words, among all upper bounds $M \in K$ of X , i.e. $x \leq M$ for every $x \in X$, $\sup X \in K$ is the least of them. Therefore, $\sup X \in K$ has the following properties:

- (i) (upper bound) For every $x \in X$, $x \leq \sup X$.
- (ii) (least upper bound) Given any $a \in K$ such that $x \leq a$ for every $x \in X$, then $\sup X \leq a$. In other words, if $a < \sup X$ then $\exists b \in X$ such that $a < b$.

Lemma 82. *If the supremum of a set X exists, it is unique.*

Proof. Suppos $a = \sup X$ and $b = \sup X$. By (ii) above, $a \leq b$ since a is the least upper bound, but for the same reason we also have $b \leq a$, hence $a = b$. \square

Lemma 83. *If a set X has a maximum element, then $\max X = \sup X$.*

Proof. Indeed, $\max X$ is obviously an upper bound and any other upper bound is greater than or equal to the maximum. \square

Example 84. *Consider the set $I_n = \{1, 2, \dots, n\} \subseteq \mathbb{Q}$. Then $\sup I_n = \max I_n = n$.*

Example 85. *Consider the set $X = \{-\frac{1}{n}; n \in \mathbb{N}\} \subseteq \mathbb{Q}$. Then $\sup X = 0$. Indeed, 0 is an upper bound and given any number $a < 0$ we can find $-\frac{1}{n}$ such that $a < -\frac{1}{n}$ since \mathbb{Q} is an Archimedean field.*

Similar to the idea of supremum, the **infimum** of a bounded from below set $X \subseteq K$, denoted $\inf X$, is the greatest lower bound. The element $\inf X \in K$ has the following properties:

- (i) (lower bound) For every $x \in X$, $x \geq \inf X$.
- (ii) (greatest lower bound) Given any $a \in K$ such that $x \geq a$ for every $x \in X$, then $\inf X \geq a$.

The lemmas 82 and 83 extend naturally to the notion of infimum, namely, if $X \subseteq K$ has a minimum element m then $m = \inf X$. Additionally, the infimum is unique. More generally, we easily conclude that:

Proposition 86. *Let $X \subseteq K$ be a bounded subset of an ordered field K . Then, $\inf X \in X \iff \inf X = \min X$ and $\sup X \in X \iff \sup X = \max X$. In particular, every finite set has a supremum and infimum.*

Example 87. *Consider the set $X = (a, b)$, an open interval in a ordered field K . Then $\inf X = a$ and $\sup X = b$. Indeed, a is a lower bound, by definition of interval, suppose $c > a$, we claim c can't be a lower bound. For instance, consider $d = \frac{a+c}{2} \in (a, b)$. We have $d < c$ if $c < b$, hence the conclusion.*

Example 88. Let $X = \{\frac{1}{2^n}; n \in \mathbb{N}\} \subseteq \mathbb{Q}$. Then $\inf X = 0$ and $\sup X = \frac{1}{2}$. Notice that $\max X = \frac{1}{2}$, by lemma 83 $\sup X = \frac{1}{2}$. Now, 0 is obviously a lower bound. Suppose $c > 0$, since \mathbb{Q} is Archimedean we can find $n \in \mathbb{N}$ such that $n + 1 > \frac{1}{c}$. By Bernoulli's inequality (Proposition 74), we have $2^n = (1 + 1)^n \geq 1 + n > \frac{1}{c}$, hence $c > \frac{1}{2^n}$ and c can't be a lower bound, so $\inf X = 0$.

Lemma 89. (Pythagoras) There is no $x \in \mathbb{Q}$ satisfying $x^2 = 2$.

Proof. Suppose not, then $x = \frac{p}{q}$ satisfies $\left(\frac{p}{q}\right)^2 = 2$, or $p^2 = 2q^2$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. If we decompose p^2 in prime factors, it will have an even number of factors equal to two, the same occurs for q^2 . Since $2q^2$ has an odd number of factors two, we can't have $p^2 = 2q^2$. \square

Proposition 90. Consider the sets of rational numbers $X = \{x \in \mathbb{Q}; x \geq 0 \text{ and } x^2 < 2\}$ and $Y = \{y \in \mathbb{Q}; y > 0 \text{ and } y^2 > 2\}$. There are no rational numbers $a, b \in \mathbb{Q}$ such that $a = \sup X$ and $b = \inf Y$.

Proof. We prove the result concerning the supremum, the result about infimum can be proven similarly. We first claim X doesn't have a maximum element. Given $x \in X$, take $r < 1$ satisfying $0 < r < \frac{2-x^2}{2x+1}$, then $x + r \in X$, so $x \in X$ can't be the maximum. Indeed, since $r < 1 \Rightarrow r^2 < r$, and we have

$$(x + r)^2 = x^2 + 2xr + r^2 < x^2 + 2xr + r = x^2 + r(2x + 1) < x^2 + 2 - x^2 = 2.$$

By a similar reasoning, given $y \in Y$, it's possible to find $r > 0$ such that $y - r \in Y$, so Y doesn't have a minimum element. Finally, notice that if $x \in X$, $y \in Y$ then $x < y$, since $x^2 < 2 < y^2 \Rightarrow 0 < (x - y)(x + y) \Rightarrow 0 < (x - y)$.

Suppose there is a number $a \in \mathbb{Q}$ such that $a = \sup X$. Then $a \notin X$, otherwise it would be its maximum. If $a \in Y$, since Y doesn't have a minimum, there would be a $b \in Y$ such that $b < a$, then $x < b < a$, a contradiction since a is the supremum. We conclude that $a \notin X$ and $a \notin Y$, so a has to satisfy $a^2 = 2$, a contradiction by lemma 89. \square

Since every ordered field contains \mathbb{Q} , in the proposition above, if there is an ordered field K such that every nonempty bounded from above set has a supremum, then $a = \sup X$ is an element of K satisfying $a^2 = 2$.

Example 91. (A bounded set with no supremum) Let K be a non-Archimedean field. Then, by definition, $\mathbb{N} \subseteq K$ is bounded from above. Let $M \in K$ be an

upper bound for \mathbb{N} . So $n + 1 \leq M$ for all $n \in \mathbb{N}$, but then $n \leq M - 1$ and $M - 1$ is also an upper bound. We conclude that if M is an upper bound, $M - 1$ is one as well, hence $\sup \mathbb{N}$ doesn't exist in K .

We say that an ordered field K is **complete**, if every nonempty bounded from above subset $X \subseteq K$ has a supremum in K . This motivates the following axiom (also called **the fundamental axiom of mathematical analysis**):

Axiom. There is a complete ordered field, represented by \mathbb{R} , called the field of real numbers.

Remark 3. Notice that in a complete ordered field K , if $X \subseteq K$ is bounded from below then X has an infimum.

Remark 4. From example 91 we conclude that every complete ordered field is Archimedean.

Proposition 92. If K, L are complete ordered fields, then there is an isomorphism $f : K \rightarrow L$.

The proposition above says that, in some suitable sense, \mathbb{R} is the only complete ordered field.

Until the end of the semester, **every** topic we discuss will involve the complete ordered field \mathbb{R} and its properties.

The discussion above leads to the conclusion that despite there is no number $x \in \mathbb{Q}$ satisfying $x^2 = 2$, there is a positive number $x \in \mathbb{R}$ such that $x^2 = 2$. We denote that number by $\sqrt{2}$. There is nothing special about 2, so we can generalize the proof above to any $n \in \mathbb{N}$ that is not a perfect square and conclude that we can find a positive number, denoted by \sqrt{n} , such that $(\sqrt{n})^2 = n$.

We can generalize even further and talk about the n^{th} -root of $m \in \mathbb{N}$, denote by $\sqrt[n]{m}$. Namely, a positive number $x \in \mathbb{R}$ such that $x^n = m$.

We call the elements of the set $\mathbb{R} - \mathbb{Q}$, **irrational numbers**. As we've just seen, there are many of them, namely, numbers of the form $\sqrt[n]{2}$, for $n \geq 2$, are all irrational. In fact, we shall see next that irrational numbers are everywhere, in a precise sense, as a subset of the real numbers.

A subset $X \subseteq \mathbb{R}$ is said to be **dense** in \mathbb{R} if for every $a, b \in \mathbb{R}$, with $a < b$, we can find $x \in X$ such that $a < x < b$. In other words, X is dense in \mathbb{R} if every open non-degenerate interval (a, b) contains a point $x \in X$.

Example 93. Let $X = \mathbb{R} - \mathbb{Z}$. Then X is dense in \mathbb{R} . Indeed, every open interval (a, b) is an infinite set (since \mathbb{R} is ordered). On the other hand, $\mathbb{Z} \cap (a, b)$ is finite, hence we can always find a number $x \notin \mathbb{Z}$ with $x \in (a, b)$.

Theorem 94. The set of rational numbers, \mathbb{Q} , and the set of irrational numbers, $\mathbb{R} - \mathbb{Q}$, are both dense in \mathbb{R} .

Proof. Let $(a, b) \in \mathbb{R}$ be a non-degenerate open interval. The idea of the proof is that since $b - a > 0$, there is a natural number $n \in \mathbb{N}$ such that $\frac{1}{n} < b - a$, hence a multiple of this number, say $\frac{m}{n}$ eventually will be in (a, b) . More formally, let $X = \{m \in \mathbb{Z}; \frac{m}{n} \geq b\}$. Since \mathbb{R} is Archimedean, $X \neq \emptyset$. Notice that X is bounded from below by $nb \in \mathbb{R}$. By the well ordering principle, X has a smallest element, say $m_0 \in X$. By the smallness of m_0 , the number $m_0 - 1 \notin X$, so $\frac{m_0 - 1}{n} < b$. We claim $a < \frac{m_0 - 1}{n}$. Suppose not, then $\frac{m_0 - 1}{n} \leq a < b < \frac{m_0}{n}$, which implies that $b - a \leq \frac{m_0}{n} - \frac{m_0 - 1}{n} = \frac{1}{n}$, a contradiction. Therefore, the rational number $\frac{m_0 - 1}{n}$ satisfies $a < \frac{m_0 - 1}{n} < b$ and \mathbb{Q} is dense in \mathbb{R} . We can apply the same argument *mutatis mutandis* to conclude that $\mathbb{R} - \mathbb{Q}$ is dense. Namely, instead of using $\frac{1}{n}$ in our argument, we use an irrational number, say $\frac{\sqrt{2}}{n}$. \square

Theorem 95. (The nested intervals principle) Let $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$ be a decreasing sequence of closed intervals of the form $I_n = [a_n, b_n]$. Then $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$, or more precisely,

$$\bigcap_{n=1}^{\infty} I_n = [a, b],$$

where $a = \sup a_n = \sup\{a_n; n \in \mathbb{N}\}$ and $b = \inf b_n = \inf\{b_n; n \in \mathbb{N}\}$

Proof. By hypothesis, $I_n \supseteq I_{n+1}, \forall n \in \mathbb{N}$, which implies:

$$a_1 \leq a_2 \leq \dots \leq a_n \leq \dots \leq b_n \leq \dots \leq b_2 \leq b_1.$$

Notice that a_n is bounded from above by b_1 , hence the supremum of a_n , $a \in \mathbb{R}$, is well defined. Similarly, the infimum of b_n , $b \in \mathbb{R}$, is well defined. Since b_n is an upper bound for a_n , we have $a \leq b_n, \forall n \in \mathbb{N}$. On the other hand, a is also an upper bound and we conclude that

$$a_n \leq a \leq b_n, \forall n \in \mathbb{N}.$$

A similar reasoning can be applied to b , hence

$$[a, b] \subseteq I_n, \forall n \in \mathbb{N}.$$

If $x < a$, we can find a_{n_0} such that $x < a_{n_0}$, so $x \notin I_{n_0} \Rightarrow x \notin \bigcap_{n=1}^{\infty} I_n$. Similarly, if $x > b$, then we can find n_1 such that $b_{n_1} < x$, so $x \notin I_{n_1} \Rightarrow x \notin \bigcap_{n=1}^{\infty} I_n$. We conclude that $\bigcap_{n=1}^{\infty} I_n = [a, b]$. \square

Theorem 96. \mathbb{R} is uncountable.

Proof. Let $X = \{x_1, x_2, \dots\} \subseteq \mathbb{R}$ be a countable subset of \mathbb{R} , which we know exists by theorem 48. We claim there is always an $x \in \mathbb{R}$ such that $x \notin X$. Pick a closed interval I_1 not containing x_1 , this is possible since \mathbb{R} is infinite. Proceed by induction, after setting I_n not containing x_n , we select $I_{n+1} \subseteq I_n$ as a closed interval which doesn't contain x_{n+1} . Proceeding this way, we construct a nested sequence of closed intervals $I_1 \supseteq I_2 \supseteq \dots \supseteq I_n \supseteq \dots$. Therefore, by theorem 95, there is at least one $x \in \mathbb{R}$ that is not in X . \square

Corollary 97. Any non-degenerate interval $(a, b) \subseteq \mathbb{R}$ is uncountable.

Proof. The function $f : (0, 1) \rightarrow (a, b)$ defined by $f(x) = (b-a)x + a$ is bijective, so it suffices to prove the result for $(0, 1)$. Suppose $(0, 1)$ is countable, then $(0, 1]$ is also countable and reasoning as before, $(n, n+1]$ is countable for every $n \in \mathbb{Z}$. Then $\mathbb{R} = \bigcup_{n \in \mathbb{Z}} (n, n+1]$ is countable, a contradiction. \square

Corollary 98. The set of irrational numbers $\mathbb{R} - \mathbb{Q}$ is uncountable.

Proof. Suppose not, then $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} - \mathbb{Q})$ is countable, a contradiction. \square

13 Sequences

A **sequence of real numbers**, denoted by $x_n := x(n)$, is a function $x : \mathbb{N} \rightarrow \mathbb{R}$ that associates to each natural number $n \in \mathbb{N}$, a real number $x(n) \in \mathbb{R}$. There is no universally defined notation for a sequence x_n , but here are examples of common notation found in the literature:

$$\{x_n\}_{n \in \mathbb{N}}, \mathbf{x}_n, \{x_1, x_2, \dots\}, (x_n)$$

We say that a sequence x_n is *bounded* if there are $a, b \in \mathbb{R}$ such that

$$a \leq x_n \leq b,$$

this is equivalent of saying that $x(\mathbb{N}) \subseteq [a, b]$, i.e. $x(n)$ is bounded as a function. A sequence is *unbounded* when is not bounded.

A sequence x_n is *bounded from above* when there is $b \in \mathbb{R}$ such that $x_n \leq b$, and *bounded from below* if there is an $a \in \mathbb{R}$ such that $a \leq x_n$. Notice that a sequence is bounded if and only if is bounded from above and below.

Let $K \subseteq \mathbb{N}$ be an infinite subset. Then K is countably infinite, let $b : \mathbb{N} \rightarrow K$, given by $k \mapsto n_k$ be a bijection. Given any sequence $x : \mathbb{N} \rightarrow \mathbb{R}$, the composition $x_{n_k} := x \circ b : K \rightarrow \mathbb{R}$ is also a sequence, called a **subsequence** of x_n .

Example 99. Let $K = \{n; n \text{ is even}\} \subseteq \mathbb{N}$ and $b(k) = 2k$. In this case, given a sequence x_n , the sequence $x_{n_k} := x_{2n}$ is a subsequence of x_n . For example, if $x_n = (-1)^n$, i.e. $\{-1, 1, -1, \dots\}$, then x_{2n} is the constant subsequence $x_{2n} = \{1, 1, 1, \dots\}$.

Notice that every subsequence x_{n_k} of a bounded sequence x_n is itself bounded by definition. We say a sequence x_n is *nondecreasing* if $x_n \leq x_{n+1}, \forall n \in \mathbb{N}$, and if the inequality is strict, i.e. $x_n < x_{n+1}$, we call x_n an *increasing* sequence. We define *nonincreasing* and *decreasing* sequences in a similar way by placing \geq ($>$) instead of \leq ($<$).

A sequence that is either nondecreasing, nonincreasing, increasing, or decreasing will be called **monotone**.

Lemma 100. A monotone sequence x_n is bounded \iff it has a bounded subsequence.

Proof. Only the converse is not obvious. Suppose x_{n_k} is a bounded monotone subsequence, say $x_{n_1} \leq x_{n_2} \leq \dots \leq b$. Given any $n \in \mathbb{N}$, we can find $n_k > n$, hence $x_n \leq x_{n_k} \leq b$. \square

Example 101. $x_n = 1$, i.e. $\{1, 1, 1, \dots\}$, is a constant, bounded, nonincreasing and nondecreasing sequence.

Example 102. $x_n = n$, i.e. $\{1, 2, 3, \dots\}$, is an unbounded increasing sequence.

Example 103. $x_n = \frac{1}{n}$, i.e. $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, is a bounded decreasing sequence, since $0 < x_n \leq 1$.

Example 104. $x_n = 1 + (-1)^n$, i.e. $\{0, 2, 0, 2, \dots\}$, is a bounded sequence that is not monotone.

Example 105. $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$ is increasing, and bounded, since $0 < x_n < 1 + 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^{n-1}} < 3$. The sequence $y_n = (1 + \frac{1}{n})^n$ is related to this sequence, since by the binomial theorem $y_n \leq x_n$, therefore y_n is also bounded, $0 < y_n < 3$.

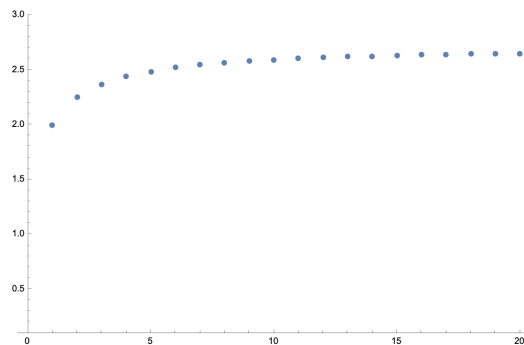


Figure 1: $y_n = (1 + \frac{1}{n})^n$

Example 106. Let $x_1 = 0$ and $x_2 = 1$, and consider, by induction, $x_{n+2} = x_{n+1} + x_n$. It's easy to see that $0 \leq x_n \leq 1$, and moreover a quick computation shows that $x_{2n} = 1 - (\frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{4^{n-1}})$ and $x_{2n+1} = \frac{1}{2} (1 + \frac{1}{4} + \frac{1}{16} + \dots + \frac{1}{4^{n-1}})$. So x_n is a bounded sequence that is not monotone.

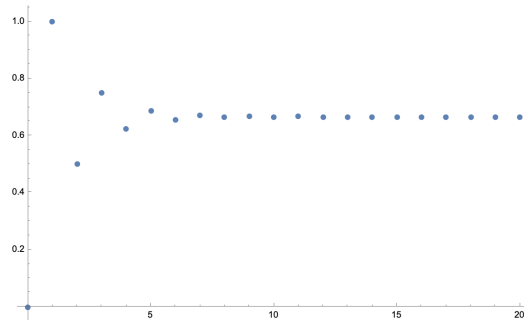


Figure 2: $x_{n+2} = x_{n+1} + x_n$

Example 107. Let $a \in \mathbb{R}$ such that $0 < a < 1$. The sequence $x_n = 1 + a + a^2 + \dots + a^n = \frac{1-a^{n+1}}{1-a}$ is increasing, since $a > 0$, and bounded since $0 < x_n \leq \frac{1}{1-a}$.

Example 108. The sequence $\{1, \sqrt{2}, \sqrt[3]{3}, \sqrt[4]{4}, \dots\}$ given by $x_n = \sqrt[n]{n}$, increases for $n = 1, 2$. We claim that starting at the third term, this sequence is decreasing. Indeed, $x_{n+1} < x_n$ is equivalent to $(n+1)^n < n^{n+1}$, which is equivalent to $(1 + \frac{1}{n})^n < n$, which is true for $n \geq 3$ by Example 105. Hence, x_n is bounded.

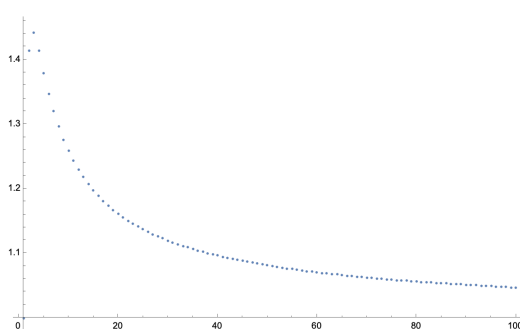


Figure 3: $x_n = \sqrt[n]{n}$

14 The limit of a sequence

Informally, to say $a \in \mathbb{R}$ is the limit of the sequence x_n is to say that the terms of the sequence are very close to a , when n is large. More precisely, we quantify this using the following definition:

$$\lim_{n \rightarrow \infty} x_n = a := \forall \epsilon > 0 \exists n_0 \in \mathbb{N}; n > n_0 \Rightarrow |x_n - a| < \epsilon$$

In other words: “The limit of sequence x_n is a , if for every positive number ϵ , no matter how small it is, it’s always possible to find an index n_0 such that the distance between x_n and a is less than ϵ , for $n > n_0$.”

Additionally, the above is the same of saying that any open interval

$$(a - \epsilon, a + \epsilon)$$

centered at a and with length 2ϵ , contains all the points of the sequence x_n except possibly a finite amount of them.

Remark 5. *It's a common practice to omit " $n \rightarrow \infty$ " and write $\lim x_n$ only.*

When $\lim x_n = a$, we say x_n converges to a , also denoted by $x_n \rightarrow a$, and call x_n convergent. If x_n is not convergent, we call it divergent, i.e. there is no $a \in \mathbb{R}$ such that $\lim x_n = a$.

Theorem 109. *(Uniqueness of the limit) If $\lim x_n = a$ and $\lim x_n = b$, then $a = b$.*

Proof. Let $\lim x_n = a$ and $b \neq a$, it's enough to prove that $\lim x_n \neq b$. Take $\epsilon = \frac{|b-a|}{2}$, then since $\lim x_n = a$, we can find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$. Therefore, $x_n \notin (b - \epsilon, b + \epsilon)$ if $n > n_0$ and we can't have $\lim x_n = b$. \square

Theorem 110. *If $\lim x_n = a$, then for every subsequence x_{n_k} of x_n , we also have $\lim x_{n_k} = a$.*

Proof. Indeed, since given $\epsilon > 0$ it's possible to find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$, the same n_0 works for x_{n_k} as well, namely, $n_k > n_0 \Rightarrow |x_{n_k} - a| < \epsilon$. \square

Corollary 111. *Let $k \in \mathbb{N}$. If $\lim x_n = a$ then $\lim x_{n+k} = a$, since x_{n+k} is a subsequence of x_n .*

In other words, Corollary 111 says that the limit of a sequence doesn't change if we omit the first k terms.

Theorem 112. *Every convergent sequence x_n is bounded.*

Proof. Suppose $\lim x_n = a$. Then it's possible to find n_0 such that $x_n \in (a - 1, a + 1)$ for $n > n_0$. Let $M = \max\{|x_1|, \dots, |x_{n_0}|, |a - 1|, |a + 1|\}$, then $x_n \in (-M, M)$. \square

Example 113. *The sequence $\{0, 1, 0, 1, 0, 1, \dots\}$ can't be convergent by theorem 110, since it has two subsequences converging to different values, namely, $x_{2n} = 1$ and $x_{2n-1} = 0$. Also, this sequence is an example of a bounded sequence which is not convergent, illustrating the fact that the converse of theorem 112 is false.*

Theorem 114. *Every bounded monotone sequence is convergent.*

Proof. Suppose $x_n \leq x_{n+1}$, the other cases are proved similarly. Since x_n is bounded, $\sup x_n$ is well defined, say $a = \sup x_n$. Let $\epsilon > 0$ be given, then $\exists n_0 \in \mathbb{N}$ such that $a - \epsilon < x_{n_0}$, but since $x_n \leq x_{n+1}$, we must have $a - \epsilon < x_n, \forall n \geq n_0$. We obviously have $x_n \leq a$, hence $a - \epsilon < x_n < a + \epsilon$ for $n > n_0$ and $\lim x_n = a$. \square

Corollary 115. *If a monotone sequence x_n has a convergent subsequence then x_n is convergent.*

Example 116. *Every constant sequence $x_n = k \in \mathbb{R}$ is convergent and $\lim x_n = k$.*

Example 117. *The sequence $\{1, 2, 3, 4, \dots\}$ is divergent because it's unbounded.*

Example 118. *The sequence $\{1, -1, 1, -1, \dots\}$ is divergent because it has two subsequences converging to different values.*

Example 119. *The sequence $x_n = \frac{1}{n}$ is convergent and $\lim x_n = 0$, since \mathbb{R} is Archimedean and given any $\epsilon > 0$ it's possible to find $n_0 \in \mathbb{N}$ such that $0 < \frac{1}{n_0} < \epsilon$. Hence, $n > n_0 \Rightarrow \frac{1}{n} < \epsilon$.*

Example 120. *Let $0 < a < 1$. The sequence $x_n = a^n$ is monotone and bounded, hence convergent. Notice that $\lim x_n = 0$ in this case.*

15 Properties of limits

Theorem 121. *Let $\lim x_n = 0$ and y_n a bounded sequence. Then*

$$\lim x_n \cdot y_n = 0.$$

Proof. Let $c > 0$ be such that $|y_n| < c$. Let $\epsilon > 0$ be given, and $n_0 \in \mathbb{N}$ a number such that $n > n_0 \Rightarrow |x_n| < \frac{\epsilon}{c}$. Then, $n > n_0 \Rightarrow |x_n y_n| < \frac{\epsilon}{c} \cdot c = \epsilon$. \square

Example 122. *Using the theorem above we have $\lim_{n \rightarrow \infty} \frac{\sin n}{n} = 0$*

Theorem 123. *Let $\lim x_n = a$ and $\lim y_n = b$. Then*

1. $\lim x_n + y_n = a + b, \lim x_n - y_n = a - b;$
2. $\lim x_n \cdot y_n = ab;$

3. If $b \neq 0$ then $\lim \frac{x_n}{y_n} = \frac{a}{b}$

Example 124. Let $a \in \mathbb{R}$ be a positive number. The sequence $x_n = \sqrt[n]{a}$ is bounded and monotone, hence converges. We claim

$$\lim \sqrt[n]{a} = 1.$$

Indeed, let $L := \lim \sqrt[n]{a}$ and consider the subsequence $y_n = x_{n(n+1)}$ then

$$L = \lim y_n = \lim a^{\frac{1}{n(n+1)}} = \lim a^{\frac{1}{n} - \frac{1}{n+1}} = \frac{\lim a^{\frac{1}{n}}}{\lim a^{\frac{1}{n+1}}} = 1$$

Example 125. Similar to the example above is the sequence $x_n = \sqrt[n]{n}$. It is bounded and monotone (starting from the third term), hence converges. We claim

$$\lim \sqrt[n]{n} = 1.$$

Let $L := \lim \sqrt[n]{n}$ and consider the subsequence $y_n = x_{2n}$ then

$$L^2 = \lim y_n \cdot y_n = \lim \sqrt[2n]{2n} = \lim \sqrt[2]{2} \sqrt[n]{n} = 1 \cdot L = L$$

Hence, $L = 0$ or $L = 1$, but $L \neq 0$ since $x_n \geq 1$.

Theorem 126. If $\lim x_n = a$ and $a > 0$, then $\exists n_0$ such that $x_n > 0$ for $n > n_0$. An equivalent statement is valid if $a < 0$, namely, up to a finite amount of indexes, $x_n < 0$.

Proof. It's possible to find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \frac{a}{2}$, in particular, $x > \frac{a}{2} > 0$ if $n > n_0$. The case $a < 0$ is proved similarly. \square

Corollary 127. If x_n, y_n are convergent sequences and $x_n \leq y_n$ then $\lim x_n \leq \lim y_n$.

Corollary 128. If x_n is convergent and $x_n \geq a \in \mathbb{R}$ then $\lim x_n \geq a$.

Theorem 129. (Squeeze theorem) If $x_n \leq y_n \leq z_n$ and $\lim x_n = \lim z_n$, then $\lim y_n = \lim x_n = \lim z_n$.

16 $\liminf x_n$ and $\limsup x_n$