# Remarks on the Hodge Conjecture for Fermat Varieties

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The Hodge conjecture is major open problem in Complex Algebraic Geometry that has been puzzling mathematician for many decades. The modern statement is the following: Let $X$ be smooth complex projective variety, then the (rational) cycle class map is surjective:

$$cl_{\otimes\mathbb{Q}} : CH^p(X) \otimes \mathbb{Q} \to H^{p,p} \cap H^{2p}(X, \mathbb{Q})$$

where $cl_{\otimes\mathbb{Q}}(\sum a_i X_i) = \sum a_i [X_i]$, $a_i \in \mathbb{Q}$ and $[X_i]$ is the class of the subvariety $X_i$.

The initial statement, made by Hodge (during the ICM 1950), was that the conjecture above should hold integrally. But this was disproved years later by Grothendieck using a product of Elliptic curves. The only case of the conjecture which holds in complete generality (and over the integers) is when $p = 1$, that is to say the integral class map is surjective:

$$cl : CH^p(X) \to H^{p,p} \cap H^{2p}(X, \mathbb{Z})$$

This is the Lefschetz's theorem on $(1, 1)$-classes, which was proved by Solomon Lefschetz using Poincare's theory of Normal functions. Roughly speaking, a primitive cohomology class is the class of a normal function and since the Abel-Jacobi map is surjective in this case, we get that every normal function comes from a divisor.

The modern proof of his theorem is just take the cohomology of the short exact sequence of sheaves:

$$0 \to \mathbb{Z} \to \mathcal{O} \xrightarrow{exp} \mathcal{O}^{\times} \to 0$$

and notice that $CH^1(X) \cong CL(X) \cong Pic(X) = H^1(X, \mathcal{O}^{\times})$ and $H^2(X, \mathcal{O}) \cong H^{0,2}(X)$.

Special cases of the conjecture have emerged during the years but all of them were specific for certain classes of varieties. For example, Abelian varieties of prime dimension, unirational and uniruled fourfolds, hypersurfaces of degree less than 6, and some others.

Using the hard Lefschetz theorem, Lefschetz hyperplane theorem, Lefschetz decomposition and some Hilbert scheme arguments, we can reduce the Hodge conjecture to the case of an even dimensional ($> 2$) variety and primitive middle cohomology classes.

Shioda gave an interesting characterization of the Hodge conjecture for Fermat varieties, which we now review.

Let $X_m^n \in \mathbb{P}^{n+1}$ denote the Fermat variety of dimension $n$ and degree $m$, i.e. the solution to the equation:

$$x_0^m + x_1^m + \ldots + x_{n+1}^m = 0$$

and $\mu_m$ the group of $m$-th roots of unity. Let $G_m^n$ be quotient of the group $\overbrace{\mu_m \times \ldots \times \mu_m}^{n+2}$ by the subgroup of diagonal elements.

The group $G_m^n$ acts naturally on $X_m^n$ by coordinatewise multiplication, moreover, the character group $\hat{G}_m^n$ of $G_m^n$ can be identified with the group:

$$\hat{G}_m^n = \{(a_0, \ldots, a_{n+1}) | a_i \in \mathbb{Z}_m, a_0 + \ldots + a_{n+1} = 0\}$$

via $(\zeta_0, \ldots, \zeta_{n+1}) \mapsto \zeta_0^{a_0} \ldots \zeta_{n+1}^{a_{n+1}}$, where $(\zeta_0, \ldots, \zeta_{n+1}) \in G_m^n$.

By the arguments above, in order to prove the Hodge conjecture, it's enough to prove it for primitive classes, therefore in this talk we will restrict our attention to primitive cohomology.

The action of $G_m^n$ extends to the primitive cohomology and makes $H^i_{prim}(X_m^n, \mathbb{Q})$ and $H^i_{prim}(X_m^n, \mathbb{C})$ a $G_m^n$-module. For $\alpha \in \hat{G}_m^n$, we set:

$$V(\alpha) = \{\xi \in H^n_{prim}(X_m^n, \mathbb{C}) | g^*(\xi) = \alpha(g)\xi \text{ for all } g \in G_m^n\}$$

The characterization of the cohomology of Fermat varieties as a direct sum of eigenspaces was first described by Katz and Ogus using the representation theory of finite groups. Let

$$\mathfrak{U}_m^n := \{\alpha = (a_0, \ldots, a_{n+1}) \in \hat{G}_m^n | a_i \neq 0 \text{ for all } i\}$$

For $\alpha \in \mathfrak{U}_m^n$ we set $|\alpha| = \sum_i \frac{<a_i>}{m}$, where $< a_i >$ is the representative of $a_i \in \mathbb{Z}_m$ between 1 and $m-1$. If $n = 2p$, we set

$$\mathfrak{B}_m^n := \{\alpha \in \mathfrak{U}_m^n || t\alpha| = p + 1 \text{ for all } t \in \mathbb{Z}_m^*\}$$

We have:

**Theorem** (Katz,Ogus) Let $Hdg^p(X_m^n) := H^{p,p} \cap H^{2p}_{prim}(X, \mathbb{Q})$ be the group of primitive Hodge cycles. Then:

(a) $\dim V(\alpha) = 0$ or $1$, and $V(\alpha) \neq 0 \iff \alpha \in \mathfrak{U}_m^n$

(b) $Hdg^p(X_m^n) = \bigoplus_{\alpha \in \mathfrak{B}_m^n} V(\alpha)$

Now let $C(X_m^n)$ denote the subspace of $Hdg^p(X_m^n)$ which are classes of algebraic cycles. Then $C(X_m^n)$ is a $G_m^n$-submodule and by the theorem above there is a subset $\mathfrak{C}_m^n \subset \mathfrak{B}_m^n$ such that:

$$C(X_m^n) = \bigoplus_{\alpha \in \mathfrak{C}_m^n} V(\alpha)$$

the Hodge conjecture can then be stated as follows:

**Conjecture** (Hodge Conjecture) For all $n, m$ we have $\mathfrak{C}_m^n = \mathfrak{B}_m^n$.

By the discussion in the previous section, this is true for $n \leq 2$ and all $m$. The idea to prove this equality for Fermat varieties it to use the fact that $X_m^n$ 'contains' disjoint unions of $X_m^k$ with $k < n$, we then blow that up to find a relation between the cohomologies and to inductively construct algebraic cycles in $X_m^n$. More precisely, we have the following result due to Shioda:

**Theorem** (Shioda) Let $n = r + s$ with $r, s \geq 1$. Then there is an isomorphism

$$[H^r_{prim}(X^r_m, \mathbb{C}) \otimes H^s_{prim}(X^s_m, \mathbb{C})]^{\mu_m} \oplus H^{r-1}_{prim}(X^{r-1}_m, \mathbb{C}) \otimes H^{s-1}_{prim}(X^{s-1}_m, \mathbb{C}$$

$$\Big\downarrow f$$

$$H^n_{prim}(X^n_m, \mathbb{C})$$

with the following properties:

a) $f$ is $G_m^n$-equivariant

b) $f$ is morphism of Hodge structures of type (0,0) on the first summand and of type $(1,1)$ on the second.

c) If $n = 2p$ then $f$ preserves algebraic cycles, moreover if

$$Z_1 \otimes Z_2 \in H^{r-1}_{prim}(X^{r-1}_m, \mathbb{C}) \otimes H^{s-1}_{prim}(X^{s-1}_m, \mathbb{C})$$

then $f(Z_1 \otimes Z_2) = mZ_1 \wedge Z_2$, where $Z_1 \wedge Z_2$ is the algebraic cycle obtained by joining $Z_1$ and $Z_2$ by lines on $X^n_m$, when $Z_1, Z_2$ are viewed as cycles in $X^n_m$.

Shioda proved the following:

**Theorem** (Shioda) If $m$ is coprime to 6 then $Hdg^1(X^2_m)$ is generated by lines.

A few years later his student proved:

**Theorem** (Aoki) If $m$ is coprime to 6 then $Hdg^1(X_m^1 \times X_m^1)$ is generated by lines.

Now taking $r = s = 2$ in the theorem and using the two theorems above we immediately get:

**Theorem** If $m$ is coprime to 6 then $Hdg^2(X_m^4)$ is generated by 2-planes.

As a corollary we have:

**Theorem** If $m$ is coprime to 6 then the Hodge conjecture is true for $X_m^4$.

In light of Shioda's theorem, we introduce the following notation:
Let $\beta = (b_0, \ldots, b_{r+1}), \gamma = (c_0, \ldots, c_{s+1})$, we set:

$$\mathfrak{U}_m^{r,s} = \{(\beta, \gamma) \in \mathfrak{U}_m^r \times \mathfrak{U}_m^s \mid b_{r+1} + c_{s+1} = 0\}$$

For $(\beta, \gamma) \in \mathfrak{U}_m^{r,s}$ we define:

$$\beta \# \gamma = (b_0, \ldots, b_r, c_0, \ldots, c_s) \in \mathfrak{U}_m^{r+s}$$

and for $\beta' = (b_0, \ldots, b_r) \in \mathfrak{U}_m^{r-1}$ and $\gamma' = (c_0, \ldots, c_s) \in \mathfrak{U}_m^{s-1}$, we set:

$$\beta' * \gamma' = (b_0, \ldots, b_r, c_0, \ldots, c_s) \in \mathfrak{U}_m^{r+s}$$

As a corollary of Shioda's theorem we have:

**Corollary** Suppose $n = 2p = r + s$, where $r, s \geq 1$.

a) If $r, s$ are odd and $(\beta', \gamma') \in \mathfrak{C}_m^{r-1} \times \mathfrak{C}_m^{s-1}$ then $\beta' * \gamma' \in \mathfrak{C}_m^n$

b) If $r, s$ are even and $(\beta, \gamma) \in (\mathfrak{C}_m^r \times \mathfrak{C}_m^s) \cap \mathfrak{U}_m^{r,s}$ then $\beta \# \gamma \in \mathfrak{C}_m^n$

Therefore, the Hodge conjecture can be proven for the Fermat $X_m^n$ if the following conditions are true for every $\alpha \in \mathfrak{B}_m^n$:

(P1) $\alpha \sim \beta' * \gamma'$ for some $(\beta', \gamma') \in \mathfrak{B}_m^{r-1} \times \mathfrak{B}_m^{s-1}, \quad (r, s \text{ odd})$.

(P2) $\alpha \sim \beta \# \gamma$ for some $(\beta, \gamma) \in (\mathfrak{B}_m^r \times \mathfrak{B}_m^s) \cap \mathfrak{U}_m^{r,s}$, (r,s even and positive).

where $\sim$ means equality up to permutation between factors.

In order to make these conditions more explicit, we introduce the additive semi-group $M_m$ of non-negative solutions $(x_1, \ldots, x_{m-1}; y)$, with $y > 0$, of the following system of linear equations:

$$\sum_{i=1}^{m-1} < ti > x_i = my \text{ for all } t \in \mathbb{Z}_m^*$$

Also, define $M_m(y)$ as those solutions where $y$ is fixed. Note that by Gordan's lemma, $M_m$ is finitely generated.

**Definition** An element $a \in M_m$ is called **decomposable** if $a = c + d$ for some $c, d \in M_m$, otherwise it's called **indecomposable**. An element is called **quasi-decomposable** if $a + b = c + d$ for some $a \in M_m(1)$ and $c, d \in M_m$ with $c, d \neq a$.

With this notation we can identify elements of $\mathfrak{B}_m^n$ with elements of $M_m$ using the map:

$$\begin{array}{c} \alpha = (a_0, \ldots, a_{n+1}) \in \mathfrak{B}_m^n \\ \Big\downarrow \{\cdot\} \\ \{\alpha\} = (x_1(\alpha), \ldots, x_{m-1}(\alpha), \frac{n}{2}+1) \in M_m(\frac{n}{2}+1) \end{array}$$

where $x_k(\alpha)$ is the number os i's such that $< a_i >= k$.

Note that $\alpha$ satisfies $(P1)$ above if and only if $\{\alpha\}$ is decomposable. If $\alpha$ satisfies $(P2)$ then $\{\alpha\}$ is quasi-decomposable. Conversely, if the latter is true then $\alpha$ satisfies $(P1)$ or $(P2)$. So it makes sense to introduce the following conditions:

($P_m^n$) Every indecomposable elemets of $M_m(y)$ with $3 \leq y \leq \frac{n}{2}+1$, if any, is quasi-decomposable.

($P_m$) Every indecomposable elemets of $M_m(y)$ with $y \geq 3$ is quasi-decomposable.

By the results above we conclude:

**Proposition** If condition ($P_m$) is satisfied, then the Hodge conjecture is true for $X_m^n$ for all $n$. If ($P_m^n$) is satisfied then the Hodge conjecture is true for $X_m^n$.

This theorem gives us a combinatorial approach to the Hodge conjecture, namely, one can check condition ($P_m^n$) or ($P_m$) and deduce from it the Hodge conjecture.

For $m \leq 20$, condition $(P_m)$ can easily be verified by hand. Also, for $m$ prime or $m = 4$, $M_m$ is generated by $M_m(1)$. In summary:

**Theorem** (Shioda, Ran) If $m \leq 20$ or $m$ prime, then the Hodge conjecture is true for $X_m^n$ for every $n$.

For $m > 20$, checking condition $(P_m)$ by hand is almost impossible for $m$ not prime. Using computer one can see that for $m = p^2$, the number of indecomposables elements are very small when compared to other values of $m$. On the other hand, some of those cases do not satisfy $(P_m)$, say $m = 25$ for example. That is not to say that Hodge conjecture is false, on the contrary, N. Aoki explicitly constructed algebraic cycles representing each $V(\alpha)$.

**Theorem** (Aoki) If $m = p^2$, then the Hodge conjecture is true for $X_m^n$ for every $n$.

Using computer we were able to verify condition ($P_m$) for $m = 21$ and $m = 27$.

**Proposition** If $m = 21$ or $m = 27$, then the Hodge conjecture is true for $X_m^n$ for every $n$.

Since 21 is a product of distinct primes, one might think that condition ($P_m$) should hold for a product of distinct primes. That is not the case:

**Proposition** Condition ($P_{33}^4$) is false.

This proposition confirms that starting at $n = 4$, there are cycles not coming from the induced structure. Therefore, we can not prove the Hodge conjecture only using this approach. One thing that can be done is to find explicitly the algebraic cycles whose class project non trivially to $V(\alpha)$ for each $\alpha \in \mathfrak{B}_m^n$.

In the particular case where $m = 3d$ and $3 \nmid d$, as above, we have a candidate. Consider the following elementary symmetric polynomials in $\mathbf{x}= (x_0, \ldots, x_5)$:

$$\begin{aligned} p_1(\mathbf{x}) &:= x_0 + x_1 + x_2 + x_3 + x_4 + x_5 \\ p_2(\mathbf{x}) &:= x_0x_1 + x_0x_2 + \ldots x_4x_5 \\ p_3(\mathbf{x}) &:= x_0x_1x_2 + \ldots x_3x_4x_5 \end{aligned} \tag{1}$$

Recall the Newton identity:

$$x_0^3+x_1^3+x_2^3+x_3^3+x_4^3+x_5^3 = p_1(\mathbf{x})^3-3p_1(\mathbf{x})p_2(\mathbf{x})+3p_3(\mathbf{x})^3 \tag{2}$$

Set $\mathbf{x}^d = (x_0^d, \dots, x_5^d)$, then:

$$x_0^m + x_1^m + x_2^m + x_3^m + x_4^m + x_5^m = p_1(\mathbf{x}^d)^3 - 3p_1(\mathbf{x}^d)p_2(\mathbf{x}^d) + 3p_3(\mathbf{x}^d)^3 \tag{3}$$

Let $W$ denotes the following variety in $\mathbb{P}^5$:

$$p_1(\mathbf{x}^d) = p_2(\mathbf{x}^d) = p_3(\mathbf{x}^d) = 0 \tag{4}$$

By construction, $W \subset X_m^4$ is a subvariety of codimension 2, so $[W] \in Hdg^2(X_m^4)$.

**Question** Can $[W]$ project non trivially in $V(\alpha)$ for every $\alpha \in \mathfrak{B}_m^4$?

If the answer is yes, then we would have a positive answer to the Hodge conjecture even if condition $(P_{3d})$ is false.

We know by Schur's lemma that the number of indecomposable elements is finite. Let $\mathcal{I}_m$ be set of indecomposable elements of $M_m$. Define $\phi : \mathbb{Z}_+ \to \mathbb{Z}_+$ by the rule

$$\phi(m) = \{\max y \mid (x_1, \dots x_{m-1}, y) \in \mathcal{I}_m\} \tag{5}$$

We have the following:

**Proposition** If the Hodge conjecture is true for $X_m^n$, for all $n \leq 2(\phi(m) - 1)$, then it's true for $X_m^n$ and any $n$.

*Proof.* The Hodge classes in $X_m^n$ are parametrized by $\mathcal{B}_m^n$, which can be viewed inside $M_m$ as elements of length $\frac{n}{2} + 1$. Since the indecomposables generate $M_m$, it is enough that those be classes of algebraic cycles. But that is the case if the Hodge conjecture is true when $\frac{n}{2} + 1 \leq \phi(m)$, by definition of $\phi(m)$. □

Therefore, for Fermat varieties of degree $m$, we don't need to check the Hodge conjecture in every dimension. It's enough to prove the result for dimension up to $2(\phi(m) - 1)$.

A natural question that arises is then what is the explicit expression of the function $\phi(m)$. For $m$ prime or $m = 4$, we know already that $\phi(m) = 1$. Also, by the work of Aoki, we know that for $p > 2$ prime $\phi(p^2) = \frac{p+1}{2}$. Here's a table with the a few values of $\phi(m)$:

| $m$ | $\phi(m)$ |
|---|---|
| 20 | 5 |
| 21 | 3 |
| 22 | 7 |
| 23 | 1 |
| 24 | 9 |
| 25 | 3 |

| $m$ | $\phi(m)$ |
|---|---|
| 26 | 7 |
| 27 | 5 |
| 28 | 7 |
| 29 | 1 |
| 30 | 9 |
| 31 | 1 |

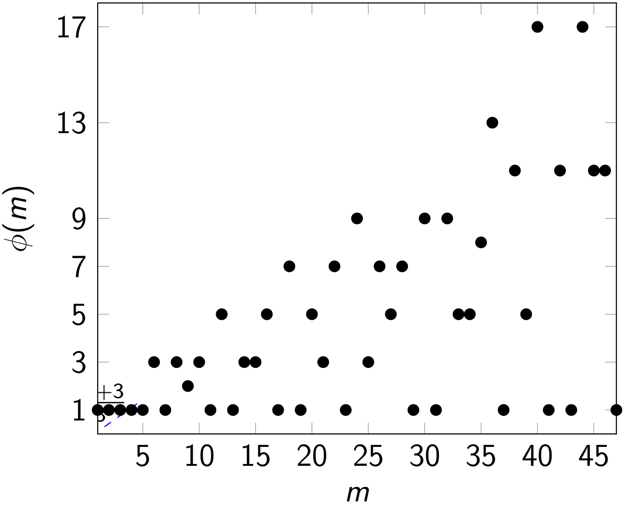
| $m$ | $\phi(m)$ |
|---|---|
| 32 | 9 |
| 33 | 5 |
| 34 | 5 |
| 35 | 8 |
| 36 | 13 |
| 37 | 1 |

| $m$ | $\phi(m)$ |
|---|---|
| 38 | 11 |
| 39 | 5 |
| 40 | 17 |
| 41 | 1 |
| 42 | 11 |
| 43 | 1 |

Based on the values above and the ones already computed, we believe the following is true:

**Conjecture** For $p > 2$ prime, we have $\phi(p^k) = \frac{p^{k-1}+1}{2}$, and $\phi(2^l) = 2^{l-2} + 1$ for $l > 2$.

Computing $\phi(m)$ for $m < 48$ gives the following:



**Question** It seems that $\phi(m) \leq f(m)$ for some linear function $f$. Can $f$ be described explicitly?

For $m \geq 48$, computations become more and more time consuming, even for the computer, and specially if $m$ has a lot of prime powers in its prime decomposition. But we hope to use the results obtained here to understand the structure of the semi-group $M_m$ and consequently, prove more cases of the Hodge conjecture.

*Thank you!*