Exercises

1. Consider the following typo in the definition of limit:

$$\forall \epsilon > 0, \exists \delta > 0, x \in X,\ 0 < |x-a| < \epsilon \Rightarrow |f(x) - L| < \delta.$$

   Show that f satisfies this condition if and only if it is bounded around each interval centered in $a \in X$. In the affirmative case, L can be any real number.

2. Let $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be given by

$$\frac{1}{1+e^{\frac{1}{x}}}.$$

   Compute $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

3. Let $f(x) = x + 10\sin x$. Show that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

4. Let $f : X \to \mathbb{R}$ be a monotone function. Show that the set of points $a \in X'$ such that $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$ is countable.

   Solution. Suppose $f(x)$ nondecreasing, i.e. $x \leq y \Rightarrow f(x) \leq f(y)$. For eaxh $a \in X'$, let $l_a = \lim_{x\to a^+} f(x) - \lim_{x\to a^-} f(x)$. The image of $f(x)$ has an interval I_a of length l_a missing (I_a could be $\emptyset$), moreover if $a \neq b$, then $I_a \cap I_b = \emptyset$, due to the monotonicity. Since the collection I_a is disjoint for each $a \in X'$, there must be only a countable collection of them that are nonempty (The function $I_a \neq \emptyset \mapsto x \in \mathbb{Q}$, for some random x, is injective.) □

5. Let $a > 1$ and $f : \mathbb{Q} \to \mathbb{R}$ given by $f(\frac{p}{q}) = a^{\frac{p}{q}}$. Show that $\lim_{x\to 0} f(x) = 1$.

6. Let $a > 1$ and $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = a^x$. Show that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = 0$

7. Let $p(x) \in \mathbb{R}[x]$ be a polynomial. If the leading coefficient is positive, show that $\lim_{x\to+\infty} p(x) = +\infty$.

8. Find the set of adherent points at 0 of the function $f : \mathbb{R} - \{0\} \to \mathbb{R}$ be given by $f(x) = \frac{\sin\left(\frac{1}{x}\right)}{1+e^{\frac{1}{x}}}$

9. If $\lim_{x\to a} f(x) = L$, show that $\lim_{x\to a} |f(x)| = |L|$, and that the set of adherent points at a is $\{L\}, \{-L\}$ or $\{-L, L\}$.

10. Given a nonempty compact set $K \subseteq \mathbb{R}$ and a point $a \in \mathbb{R}$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ whose the set of adherent points at a is K.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by

$$f(x) = \begin{cases} x, \; x \notin \mathbb{Q} \\ 0, \; x = 0 \\ q, \; x = \frac{p}{q} \text{ and } gcd(p, q) = 1, p > 0 \end{cases}$$

Show that f is unbounded in any non-degenerate interval.

12. Recall that the floor function $\lfloor x \rfloor : \mathbb{R} \to \mathbb{Z}$ is given by $\lfloor x \rfloor :=$ largest integer less than or equal to x. Show that if $a, b \in R$ are positive numbers then

$$\lim_{x \to 0^+} \frac{x}{a} \left\lfloor \frac{b}{x} \right\rfloor = \frac{b}{a} \text{ and } \lim_{x \to 0^+} \frac{b}{x} \left\lfloor \frac{x}{a} \right\rfloor = 0$$

13. Let $f, g : X \to \mathbb{R}$ be functions bounded in a neighborhood of $a \in X'$. Show that

$$\limsup_{x \to a} (f + g) \leq \limsup_{x \to a} f(x) + \limsup_{x \to a} g(x),$$

and also that

$$\limsup_{x \to a} (-f(x)) = -\liminf_{x \to a} f(x)$$

14. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + ax \sin(x)$. Show that

$$|a| < 1 \Rightarrow \lim_{x \to \pm\infty} f(x) = \pm\infty$$