Exercises

1. If $\lim x_n = a$, show that $\lim |x_n| = |a|$. Show that the converse can be false by giving a counter example.

2. Suppose $\lim x_n = 0$. Let $y_n = \min\{|x_1|, |x_2|, \dots, |x_n|\}$. Show that $\lim y_n = 0$.

3. If $\lim x_{2n} = a$ and $\lim x_{2n-1} = a$, show that $\lim x_n = a$.

4. Given an example of a sequence x_n and a infinite decomposition of $\mathbb{N} = \mathbb{N}_1 \cup \dots \cup \mathbb{N}_k \cup \dots$, such that for every $k \in \mathbb{N}$, the subsequence $(x_n)_{n \in \mathbb{N}_k}$ has limit $a \in \mathbb{R}$ but $\lim x_n \neq a$.

5. If $\lim x_n = a$ and $\lim(x_n - y_n) = 0$, show that $\lim y_n = a$.

6. Show that $(1 - \frac{1}{n})^n$ is increasing. *Hint: Use the inequality of arithmetic and geometric means involving the* $n + 1$ *numbers* $1 - \frac{1}{n}, \dots, 1 - \frac{1}{n}, 1$.

7. Let $x_n = (1 + \frac{1}{n})^n, y_n = (1 - \frac{1}{n+1})^{n+1}$. Show that $\lim x_n y_n = 1$ and conclude that $\lim(1 - \frac{1}{n})^n = e^{-1}$.

8. Let $a \geq 0, b \geq 0$. Show that $\lim \sqrt[n]{a_n + b_n} = \max\{a, b\}$

9. Let x_n be a bounded sequence. If $\lim a_n = a$ and a_n is an accumulation point of x_n, then a is an accumulation point of x_n.

10. Let x_n, y_n be bounded sequences. Set

$$a = \liminf x_n, A = \limsup x_n, b = \liminf y_n, B = \limsup y_n$$

Show that:

a) $\limsup(x_n + y_n) \leq A + B$ and $\liminf(x_n + y_n) \geq a + b$;

b) $\limsup -x_n = -a$ and $\liminf -x_n = -A$;

c) If $x_n \geq 0, y_n \geq 0$, then $\limsup(x_n \cdot y_n) \leq A \cdot B$ and $\liminf(x_n \cdot y_n) \geq a \cdot b$.

11. For each $n \in \mathbb{N}$, let $0 \leq t_n \leq 1$. If $\lim x_n = \lim y_n = a$, show that

$$\lim[t_n x_n + (1 - t_n) y_n] = a$$

12. Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{n}$. Show that x_n is bounded and find $\lim x_n$.

13. Show that x_n doesn't have a convergent subsequence if and only if $\lim |x_n| = +\infty$.

14. Let $y_n > 0$ for every $n \in \mathbb{N}$, such that $\sum y_n = +\infty$. If x_n is a sequence such that $\lim \frac{x_n}{y_n} = a$, show that $\lim \frac{x_1 + \ldots + x_n}{y_1 + \ldots + y_n} = a$.

15. Let y_n be an increasing sequence and $\lim y_n = +\infty$. Show that

$$\lim \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = a \Rightarrow \lim \frac{x_n}{y_n} = a$$

16. Show that
$$\lim \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \frac{1}{p+1}$$

17. Show that for every $n \in \mathbb{N}$, $0 < e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}\right) < \frac{1}{n!n}$. Conclude that $e \notin \mathbb{Q}$.

18. Show that $\lim \frac{1}{2} \sqrt[n]{(n+1)(n+2)\ldots 2n} = \frac{4}{e}$.

19. Suppose the sequence x_n satisfies $n! = n^n e^{-n} x_n$. Show that $\lim \sqrt[n]{x_n} = 1$.

20. Let $\sum a_n$ and $\sum b_n$ be series with positive elements. Show that if $\sum b_n = +\infty$ and $\exists n_0 \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n}$ for $n > n_0$, then $\sum a_n = +\infty$.

21. Let $p(x) \in \mathbb{R}[x]$ be a polynomial of degree 2 or more. Show that the series $\sum \frac{1}{p(n)}$ converges.

22. If $|x| < 1$ show that $\lim\limits_{n \to \infty} \binom{m}{n} x^n = 0$ for every $m \in \mathbb{R}$, where $\binom{m}{n} := \frac{m(m-1)\ldots(m-n+1)}{n!}$.

23. Let $a \in \mathbb{R}$. Show that the series $\sum\limits_{n=0}^{\infty} \frac{a^2}{(1+a^2)^n}$ converges and find its sum.

24. Show that for every fixed $p \in \mathbb{R}$, the series $\sum \frac{1}{n(n+1)\ldots(n+p)}$ converges.

25. If $\sum a_n$ converges and $a_n > 0$ then $\sum a_n^2$ and $\frac{a_n}{1+a_n}$ also converge.

26. If $\sum a_n^2$ converges then $\frac{a_n}{n}$ also converges.

27. If a_n is decreasing and $\sum a_n$ converges then $\lim a_n \cdot n = 0$.

28. If a_n is nonincreasing with $\lim a_n = 0$, show that $\sum a_n$ converges if and only if $\sum 2^n \cdot a_{2^n}$ converges.

29. Show that the set of accumulation points of the sequence $x_n = \cos n$ is the closed interval $[-1, 1]$.

30. Let $a_1 \geq a_2 \geq \ldots \geq 0$ and $s_n = a_1 - a_2 + \ldots + (-1)^{n-1} a_n$. Show that s_n is bounded and
$$\limsup s_n - \liminf s_n = \lim a_n$$