9.57. Let $S = \mathbf{Z}$ and $T = \{4k : k \in \mathbf{Z}\}$. Thus, T is a nonempty subset of S.

(a) Prove that T is closed under addition and multiplication.
(b) If $a \in S - T$ and $b \in T$, is $ab \in T$?
(c) If $a \in S - T$ and $b \in T$, is $a + b \in T$?
(d) If $a, b \in S - T$, is it possible that $ab \in T$?
(e) If $a, b \in S - T$, is it possible that $a + b \in T$?

9.58. Prove that the multiplication in $\mathbf{Z}_n$, $n \geq 2$, defined by $[a][b] = [ab]$ is well-defined. (See Result 4.11.)

9.59. (a) Let $[a], [b] \in \mathbf{Z}_8$. If $[a] \cdot [b] = [0]$, does it follow that $[a] = [0]$ or $[b] = [0]$?
(b) How is the question in (a) answered if $\mathbf{Z}_8$ is replaced by $\mathbf{Z}_9$? by $\mathbf{Z}_{10}$? by $\mathbf{Z}_{11}$?
(c) For which integers $n \geq 2$ is the following statement true? (You are only asked to make a conjecture, not to provide a proof.) Let $[a], [b] \in \mathbf{Z}_n$, $n \geq 2$. If $[a] \cdot [b] = [0]$, then $[a] = [0]$ or $[b] = [0]$.

9.60. For integers $m, n \geq 2$ consider $\mathbf{Z}_m$ and $\mathbf{Z}_n$. Let $[a] \in \mathbf{Z}_m$ where $0 \leq a \leq m - 1$. Then $a, a + m \in [a]$ in $\mathbf{Z}_m$. If $a, a + m \in [b]$ for some $[b] \in \mathbf{Z}_n$, then what can be said of m and n?

9.61. (a) For integers $m, n \geq 2$ consider $\mathbf{Z}_m$ and $\mathbf{Z}_n$. If some element of $\mathbf{Z}_m$ also belongs to $\mathbf{Z}_n$, then what can be said of $\mathbf{Z}_m$ and $\mathbf{Z}_n$?
(b) Are there examples of integers $m, n \geq 2$ for which $\mathbf{Z}_m \cap \mathbf{Z}_n = \emptyset$?

Chapter 9 Supplemental Exercises

The Chapter Presentation for Chapter 9 can be found at `goo.gl/Tch7Cf`

9.62. Prove or disprove:

(a) There exists an integer a such that $ab \equiv 0 \pmod 3$ for every integer b.
(b) If $a \in \mathbf{Z}$, then $ab \equiv 0 \pmod 3$ for every $b \in \mathbf{Z}$.
(c) For every integer a, there exists an integer b such that $ab \equiv 0 \pmod 3$.

9.63. A relation R is defined on $\mathbf{R}$ by $a\,R\,b$ if $a - b \in \mathbf{Z}$. Prove that R is an equivalence relation and determine the equivalence classes $[1/2]$ and $[\sqrt{2}]$.

9.64. A relation R is defined on $\mathbf{Z}$ by $a\,R\,b$ if $|a - 2| = |b - 2|$. Prove that R is an equivalence relation and determine the distinct equivalence classes.

9.65. Let k and ℓ be integers such that $k + \ell \equiv 0 \pmod 3$ and let $a, b \in \mathbf{Z}$. Prove that if $a \equiv b \pmod 3$, then $ka + \ell b \equiv 0 \pmod 3$.

9.66. State and prove a generalization of Exercise 9.65.

9.67. A relation R is defined on $\mathbf{Z}$ by $a\,R\,b$ if $3 \mid (a^3 - b)$. Prove or disprove the following:
(a) R is reflexive. (b) R is transitive.

9.68. A relation R is defined on $\mathbf{Z}$ by $a\,R\,b$ if $a \equiv b \pmod 2$ and $a \equiv b \pmod 3$. Prove or disprove: R is an equivalence relation on $\mathbf{Z}$.

9.69. A relation R is defined on $\mathbf{Z}$ by $a\,R\,b$ if $a \equiv b \pmod 2$ or $a \equiv b \pmod 3$. Prove or disprove: R is an equivalence relation on $\mathbf{Z}$.

9.70. Determine each of the following.
(a) $[4]^3 = [4][4][4]$ in $\mathbf{Z}_5$ (b) $[7]^5$ in $\mathbf{Z}_{10}$

9.71. Let $S = \{(a, b) : \ a, b \in \mathbf{R}, a \neq 0\}$.

(a) Show that the relation R defined on S by $(a, b)\,R\,(c, d)$ if $ad = bc$ is an equivalence relation.

(b) Describe geometrically the elements of the equivalence classes $[(1, 2)]$ and $[(3, 0)]$.

9.72. In Exercise 9.19, a relation R was defined on $\mathbf{Z}$ by $x\,R\,y$ if $x \cdot y \geq 0$, and we were asked to determine which of the properties reflexive, symmetric and transitive are satisfied.

(a) How would our answers have changed if $x \cdot y \geq 0$ was replaced by: (i) $x \cdot y \leq 0$, (ii) $x \cdot y > 0$, (iii) $x \cdot y \neq 0$, (iv) $x \cdot y \geq 1$, (v) $x \cdot y$ is odd, (vi) $x \cdot y$ is even, (vii) $xy \not\equiv 2 \pmod 3$?

(b) What are some additional questions you could ask?

9.73. For the following statement S and proposed proof, either (1) S is true and the proof is correct, (2) S is true and the proof is incorrect or (3) S is false (and the proof is incorrect). Explain which of these occurs.

S: Every symmetric and transitive relation on a nonempty set is an equivalence relation.

Proof Let R be a symmetric and transitive relation defined on a nonempty set A. We need only show that R is reflexive. Let $x \in A$. We show that $x\,R\,x$. Let $y \in A$ such that $x\,R\,y$. Since R is symmetric, $y\,R\,x$. Now $x\,R\,y$ and $y\,R\,x$. Since R is transitive, $x\,R\,x$. Thus, R is reflexive. ■

9.74. Evaluate the proposed proof of the following result.

Result A relation R is defined on $\mathbf{Z}$ by $a\,R\,b$ if $3 \mid (a + 2b)$. Then R is an equivalence relation.

Proof Assume that $a\,R\,a$. Then $3 \mid (a + 2a)$. Since $a + 2a = 3a$ and $a \in \mathbf{Z}$, it follows that $3 \mid 3a$ or $3 \mid (a + 2a)$. Therefore, $a\,R\,a$ and R is reflexive.

Next, we show that R is symmetric. Assume that $a\,R\,b$. Then $3 \mid (a + 2b)$. So, $a + 2b = 3x$, where $x \in \mathbf{Z}$. Hence, $a = 3x - 2b$. Therefore,

$$b + 2a = b + 2(3x - 2b) = b + 6x - 4b = 6x - 3b = 3(2x - b).$$

Since $2x - b$ is an integer, $3 \mid (b + 2a)$. So, $b\,R\,a$ and R is symmetric.

Finally, we show that R is transitive. Assume that $a\,R\,b$ and $b\,R\,c$. Then $3 \mid (a + 2b)$ and $3 \mid (b + 2c)$. So, $a + 2b = 3x$ and $b + 2c = 3y$, where $x, y \in \mathbf{Z}$. Adding, we have $(a + 2b) + (b + 2c) = 3x + 3y$. So,

$$a + 2c = 3x + 3y - 3b = 3(x + y - b).$$

Since $x + y - b$ is an integer, $3 \mid (a + 2c)$. Hence, $a\,R\,c$ and R is transitive. ■

9.75. (a) Show that the relation R defined on $\mathbf{R} \times \mathbf{R}$ by $(a, b)\,R\,(c, d)$ if $|a| + |b| = |c| + |d|$ is an equivalence relation.

(b) Describe geometrically the elements of the equivalence classes $[(1, 2)]$ and $[(3, 0)]$.

9.76. Let $x \in \mathbf{Z}_m$ and $y \in \mathbf{Z}_n$, where $m, n \geq 2$. If $x \subseteq y$, then what can be said of m and n?

9.77. Let A be a nonempty set and let B be a fixed subset of A. A relation R is defined on $\mathcal{P}(A)$ by $X\,R\,Y$ if $X \cap B = Y \cap B$.

(a) Prove that R is an equivalence relation.

(b) Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 4\}$. For $X = \{2, 3, 4\}$, determine $[X]$.

9.78. Let R_1 and R_2 be equivalence relations on a nonempty set A. Prove or disprove each of the following.

(a) If $R_1 \cap R_2$ is reflexive, then so are R_1 and R_2.

(b) If $R_1 \cap R_2$ is symmetric, then so are R_1 and R_2.

(c) If $R_1 \cap R_2$ is transitive, then so are R_1 and R_2.

9.79. Prove that if R is an equivalence relation on a set A, then the inverse relation R^{-1} is an equivalence relation on A.

9.80. Let R_1 and R_2 be equivalence relations on a nonempty set A. A relation $R = R_1R_2$ is defined on A as follows: For $a, b \in A$, $a\ R\ b$ if there exists $c \in A$ such that $a\ R_1\ c$ and $c\ R_2\ b$. Prove or disprove: R is an equivalence relation on A.

9.81. A relation R on a nonempty set S is called **sequential** if for every sequence x, y, z of elements of S (distinct or not), at least one of the ordered pairs (x, y) and (y, z) belongs to R. Prove or disprove: Every symmetric, sequential relation on a nonempty set is an equivalence relation.

9.82. Consider the subset $H = \{[3k] : k \in \mathbf{Z}\}$ of $\mathbf{Z}_{12}$.

(a) Determine the distinct elements of H and construct an addition table for H.

(b) A relation R on $\mathbf{Z}_{12}$ is defined by $[a]\ R\ [b]$ if $[a - b] \in H$. Show that R is an equivalence relation and determine the distinct equivalence classes.

9.83. For elements $a, b \in \mathbf{Z}_n$, $n \geq 2$, $a = [c]$ and $b = [d]$ for some integers c and d. Define $a - b = [c] - [d]$ as the equivalence class $[c - d]$. Let $H = \{x_1, x_2, \ldots, x_d\}$ be a subset of $\mathbf{Z}_n$, $n \geq 2$, such that a relation R defined on $\mathbf{Z}_n$ by $a\ R\ b$ if $a - b \in H$ is an equivalence relation.

(a) For each $a \in \mathbf{Z}_n$, determine the equivalence class $[a]$ and show that $[a]$ consists of d elements.

(b) Prove that $d \mid n$.