Exercises

1. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that the *zero set* of f

$$Z(f) = \{x; f(x) = 0\}$$

is a closed set. Conclude that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous then the zero set $\{x; f(x) = g(x)\}$ is closed.

Solution. Take $x_n \in Z(f)$ such that $x_n \to a$. By continuity, $f(x_n) \to f(a)$, but $f(x_n) = 0$, hence $f(a) = 0$. We conclude that $a \in Z(f)$, which implies $\overline{Z(f)} = Z(f)$. □

2. Let $f : X \to \mathbb{R}$ be continuous. Show that for every $k \in \mathbb{R}$, the set of all $x \in X$ such that $f(x) \leq k$ is of the form $C \cap X$, where C is closed.

Solution. It suffices to prove that $f^{-1}((-\infty, k])$ is closed, or equivalently that $\mathbb{R} - f^{-1}((-\infty, k])$ is open. Notice that

$$\mathbb{R} - f^{-1}((-\infty, k]) = f^{-1}(\mathbb{R} - (-\infty, k]) = f^{-1}[(k, +\infty)]$$

is open by question 4 below. □

3. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the sets $\{x; f(x) < c\}$ and $\{x; f(x) > c\}$ are open for every $c \in \mathbb{R}$.

4. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ an open set. Show that f is continuous if and only if the set $f^{-1}(A)$ is open for every open $A \subseteq \mathbb{R}$.

Solution. Suppose f continuous. Take $b \in f^{-1}(A)$, i.e. $f(b) = a$ for some $a \in A$. Since f is continuous, it follows that $\lim_{x\to b} f(x) = a$. Take $\epsilon > 0$ small enough such that $|f(x) - a| < \epsilon \Rightarrow f(x) \in A$, which is possible since A is open. By continuity, we can find $\delta > 0$, such that $|x - b| < \delta \Rightarrow f(x) \in A$, therefore $(x - \delta, x + \delta) \subseteq f^{-1}(A)$, hence $f^{-1}(A)$ is open. Conversely, suppose the inverse image of an open set is open. Let $a \in X$ and $\epsilon > 0$ be given. By hypothesis, the set $J := f^{-1}[(-\infty, f(a) + \epsilon)] \cap f^{-1}[(f(a) - \epsilon, +\infty)]$ is open and nonempty since $a \in J$, so we can find an interval $(a - \delta, a + \delta) \subseteq J$, which implies that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. □

5. Let $f : X \to \mathbb{R}$ be a function and $X \subseteq \mathbb{R}$ a closed set. Show that f is continuous if and only if the set $f^{-1}(C)$ is closed for every closed set $C \subseteq \mathbb{R}$.

Solution. The result follows directly from question 4 above since

$$\mathbb{R} - f^{-1}(C) = f^{-1}(\mathbb{R} - C)$$

□

6. Let $S \subseteq \mathbb{R}$ be nonempty. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \inf\{|x - s|; s \in S\}$$

Show that f is *Lipschitz*: $\forall x, y \in \mathbb{R} \Rightarrow |f(x) - f(y)| \leq |x - y|$.

Solution. Fix $s \in S$. We have $f(x) \leq |x-s| \leq |x-y| + |y-s|$, hence $f(x) - |y-s| \leq |x-y|$, taking inf on both sides gives $f(x) - f(y) \leq |x-y|$. We can similarly obtain $f(y) - f(x) \leq |x-y|$, hence $|f(x) - f(y)| \leq |x-y|$. □

7. Let $X \subseteq \mathbb{R}$ be a closed set and $f : X \to \mathbb{R}$ continuous. Show that there exist a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g_{|X} = f$.

Solution. The case $X = \emptyset$ is trivial, suppose $X \neq \emptyset$. By hypothesis $X^c = \mathbb{R} - X$ is open, in particular it can be written as a countable union of disjoints open intervals $X^c = \bigcup_{n=1}^{\infty} (a_n, b_n)$. If $a_n = -\infty$ for some n, set $g(x) = f(b_n)$ for $x \in (-\infty, b_n)$, similarly, if $b_n = +\infty$ then set $g(x) = f(a_n)$ for $x \in (a_n, +\infty)$. Otherwise, set $g(x) = (1-t)f(a_n) + tf(b_n)$, for $x \in (a_n, b_n)$ and $t = \frac{x - a_n}{b_n - a_n}$. □

8. Give an example of a bijective function $f : \mathbb{R} \to \mathbb{R}$ which is discontinuous at every $a \in \mathbb{R}$.

Solution. $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = x + 1$ otherwise. □

9. Show that there is no continuous function $f : \mathbb{R} \to \mathbb{R}$ that takes every rational number to an irrational number, and vice-versa.

Solution. Suppose there is such a function. We know that if f is continuous, it takes intervals to intervals. In particular, $f(\mathbb{R})$ should be an interval, since $\mathbb{R} = (-\infty, +\infty)$. However, $f(\mathbb{R}) = f(\mathbb{Q} \cup \mathbb{Q}^c) = f(\mathbb{Q}) \cup f(\mathbb{Q}^c) \subseteq f(\mathbb{Q}) \cup \mathbb{Q}$, a contradiction since $f(\mathbb{Q}) \cup \mathbb{Q}$ is countable, hence can't contain an interval (which is uncountable). □

10. Let A be the set of all nonnegative algebraic numbers, and B be the set of negative transcendental numbers. Let $f : A \cup B \to [0, +\infty)$ be a function defined by $f(x) = x^2$. Show that f is a continuous bijection, whose inverse f^{-1} is discontinuous at every point, except zero.

Solution. We easily can check that f is a continuous bijection. If f^{-1} were continuous then it would take $(0, +\infty)$ to an interval, but $A \cup B$ doesn't have any intervals. □

11. (Brouwer Fixed Point Theorem) Let $f : [a, b] \to [a, b]$ be a continuous function. Show that there exists a point $x \in [a, b]$ such that $f(x) = x$. *[We call such point a 'fixed point'.]*

Solution. Set $g(x) = f(x) - x$. Then since $f(a) \geq a$, we have $g(a) \geq 0$. Similarly, $f(b) \leq b \Rightarrow g(b) \leq 0$. The result follows from the intermediate value theorem. $\square$

12. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. If for every open set $A \subseteq \mathbb{R}$, the image $f(A)$ is open, then f is injective, hence monotone.

    Solution. Suppose a, b are distinct points but $f(a) = f(b) = C$. Set $g(x) = (f(x)-C)^2$, then g is also continuous and sends open sets to open sets. Moreover, $g[(a, b)] = [0, d)$ for some $d \in \mathbb{R}$, a contradiction. $\square$

13. Fix $X \subseteq \mathbb{R}$. If every continuous function defined on X is bounded then X is compact.

    Solution. The function $f : X \to X$ given by $f(x) = x$ is obviously continuous, hence $X = f(X)$ is bounded. Suppose X not closed, then we can find a sequence $x_n \in X$ such that $x_n \to a$ and $a \notin X$. The function $f(x) = \frac{1}{x-a}$ is continuous but unbounded, a contradiction. $\square$

14. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to +\infty} f(x) = +\infty$. Then f achieves its minimum value, i.e. there is $a \in \mathbb{R}$ such that $f(a) \leq f(x), \forall x \in \mathbb{R}$.

    Solution. Take $M > 0$ large enough such that there is $n \in \mathbb{N}$, $|x| > n \Rightarrow f(x) > M > f(0)$. Consider f restricted to $[-n, n]$, by Weierstrass extreme value theorem, f achieves its minimum in $[-n, n]$, since $f(x) > f(0)$ is $x \notin [-n, n]$, f achieves its minimum in $\mathbb{R}$. $\square$

15. Show that $f : (-1, 1) \to \mathbb{R}$ given by $f(x) = \frac{x}{1-|x|}$ is a homeomorphism.

    Solution. f is clearly continuous. Suppose $f(x) = f(y)$ then

    $$\frac{x}{1-|x|} = \frac{y}{1-|y|} \iff x(1-|y|) = y(1-|x|) \iff x - x|y| = y - y|x| \iff x = y,$$

    and f is injective. Given $c \in \mathbb{R}$, pick x such that $c(1-|x|) = x$, equivalently, if $c \geq 0$ take $x = \frac{c}{1+c}$, and if $c < 0$, take $x = \frac{c}{1-c}$. In either case, $f(x) = c$, and f is surjective. The inverse is given by: $f^{-1}(x) = \frac{x}{1+|x|}$, which is continuous by a mutatis mutandis argument. $\square$

16. Classify all intervals of $\mathbb{R}$ up to homeomorphism. For example, all open intervals, whether or not bounded, are homeomorphic, hence should represent the same object.

Solution. Let X be the set of all intervals I in $\mathbb{R}$ up to homeomorphism, we claim

$$X = \{(0,1), [0,1], [0,1)\}.$$

Indeed, if I is open then it is homeomorphic to $(0,1)$. If I is closed and bounded then it is homeomorphic to $[0,1]$. If I is closed and unbounded or I is half-open then it is homeomorphic to $[0,1)$. □

17. Show that the inverse of f given in exercise 15, is uniformly continuous. (Notice that f isn't)

    Solution. The inverse is given by: $f^{-1}(x) = \frac{x}{1+|x|}$. A quick computation shows that $|f'(x)| \leq 1$, hence f is Lipschitz. □

18. Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sin x$ is uniformly continuous, but $g(x) = \sin x^2$ isn't.

19. Show that a polynomial $p : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if and only if has degree at most one.

20. Show that $f(x) = x^n$ is Lipschitz in any bounded set. Moreover, prove that if $n > 1$ and f is defined on an unbounded interval, then f is not even uniformly continuous.

21. Give an example of sets A, B open and a continuous function $f : A \cup B \to \mathbb{R}$ such that $f_{|A}, f_{|B}$ are uniformly continuous but f is not.

22. Given a function $f : X \to \mathbb{R}$. Suppose that for every $\epsilon > 0$, there exists $g : X \to \mathbb{R}$ continuous, such that $\forall x \in X$, $|f(x) - g(x)| < \epsilon$. Show that f is continuous.