The Hodge Conjecture

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In this talk I'll talk about one of the most celebrated open problems in complex algebraic geometry, the Hodge Conjecture.

We will discuss first the history of the problem, the major actors involved in its formulation and some ideas introduced to solve the conjecture.

Then we'll talk about the current status of the problem and discuss (somewhat) recent techniques.

Riemann & Co

Some (Brief) History...

Although algebraic geometry as subject dates back to the Hellenistic Greek Mathematicians, the notion of Riemann surface and more generally complex algebraic varieties is more of a recent topic.

The XIX century was an important one in terms of great mathematicians working in the field of complex analysis/algebraic geometry. B. Riemann with the introduction of Riemann Surface, N. Abel and his proof of Abel's theorem, C. Jacobi and his of study of the Abel-Jacobi map, A. Legendre and his work on the Legendre family and monodromies, and more importantly, H. Poincare and his introduction of normal functions.

Example of Riemann Surface

Some (Brief) History...

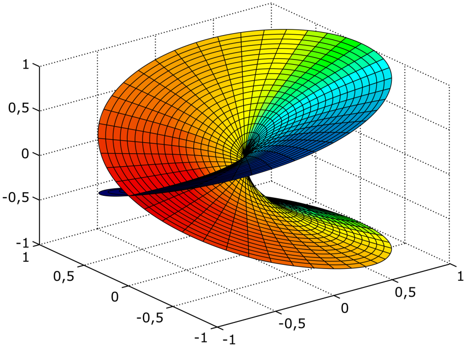
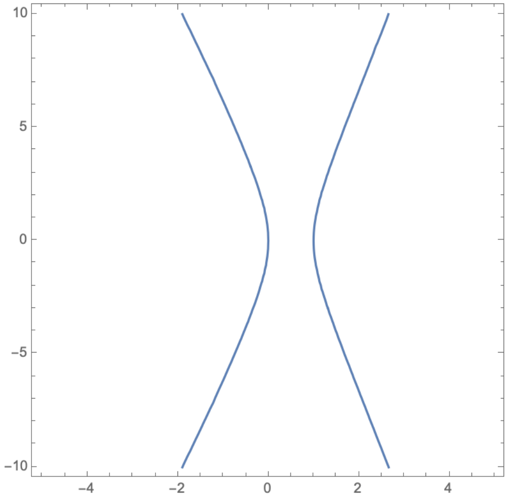


Figure: Real points of the (non-compact) Riemann surface associated to $f(z) = \sqrt{z}$.
Compact Riemann surfaces are actually algebraic curves!

The (real points of the) Legendre family

Some (Brief) History...



Lefschetz and his 1-1 theorem

Some (Brief) History...

Motivated by Poincare's study of normal functions (in 1910/11), S. Lefschetz use these functions to prove (in 1924) his famous result on the relationship between divisors and topological cycles:[1]

Theorem (Lefschetz's theorem on (1,1)-classes): The map which relates a divisor D of a smooth complex projective variety X and its cohomology class is surjective:

$$c : CH^1(X) \twoheadrightarrow H^{1,1}(X, \mathbb{Z})$$

In a few words, given a divisor D on X, let L_D be the associated line bundle, then the map c above is induced by the first Chern class map of L_D.

[1] Lefschetz used Cycle Class Group instead of Chow group, the latter were defined much later.

More explicitly, let $D = \sum a_i E_i$, where E_i are codimension 1 subvarieties. Then the map above is induced by integration over the smooth locus of E_i, i.e.

$$c(D) = a_i \int_{E_i^*} (\cdot)$$

That is to say that c is in the dual of $H^{2n-2}(X, \mathbb{C})$, by Poincare duality, this is $H^2(X, \mathbb{C})$[2].

[2] By Hodge-Riemann relations, it actually lands into $H^{1,1}(X)$ and by integral Poincare duality, we can assume integral coefficients...

Riemann surface with divisors

Some (Brief) History...

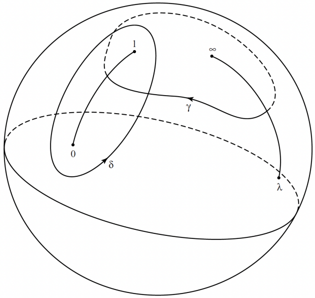


Figure: Genus-0 curve (Riemann Sphere) with cycles, divisors and branch cuts. Identifying the cuts we get a new curve (Riemann Surface) of genus 1 and a distinguished (co)homology basis.

The man himself

Some (Brief) History...

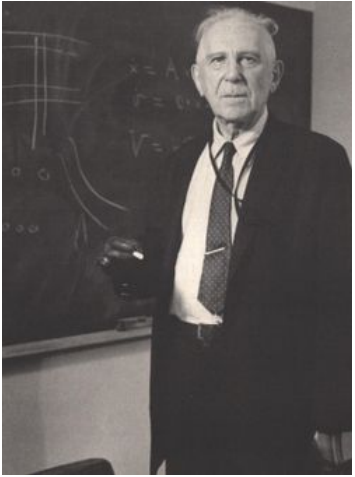


Figure: Solomon Lefschetz. An interesting fact about him is that he was responsible for the Annals of Mathematics to become a top journal. Today the journal publish almost exclusively solutions to very hard old problems/conjectures.

Example of a Lefschetz's family

Some (Brief) History...

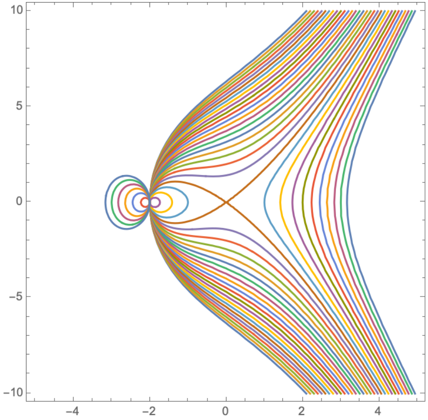


Figure: Family of curves that Lefschetz used to prove his famous theorem

Hodge's initial conjecture

Some (Brief) History...

In 1950, at the International Congress of Mathematicians in Cambridge, MA. W. Hodge inquired whether or not Lefschetz's result generalizes to higher dimensions, more precisely he stated (using modern notation)[3]:

Conjecture (Hodge (p,p)-Conjecture): The cycle class map is surjective:

$$[\cdot] : CH^p(X) \twoheadrightarrow H^{p,p}(X, \mathbb{Z})$$

More explicitly, given an algebraic cycle $D = \sum a_i D_i$ on X, the map above is induced by integration over (smooth locus of) D_i.

[3] Chow groups were only introduced in late 50's but we will use them instead for the sake of completeness.

Sir William Vallance Douglas Hodge

Some (Brief) History...

Figure: W. Hodge. He began his career working on the work of Lefschetz. Lefschetz himself was a little bit skeptical of Hodge's methods.

The corrected version

Some (Brief) History...

As stated above, this conjecture is false as proved (1961) by M. Atiyah *(one of the gentleman in the cover)* and F. Hirzebruch. They constructed a torsion class with no associated algebraic cycle. What nowadays is called Hodge conjecture is usually the following

Conjecture (Rational Hodge (p,p)-Conjecture)

$$[\cdot] : CH^p(X, \mathbb{Q}) \twoheadrightarrow H^{p,p}(X, \mathbb{Q})$$

Apart from codimension 1 and dimension up to 3 for trivial reasons, this conjecture is still open starting at dimension 4.

Another conjecture by Hodge

Some (Brief) History...

Hodge went further and asked whether or not an even stronger conjecture should be true. We say that a cohomology cycle has coniveau p if it's supported on a codimension p-subvariety. The cohomology then can be filtered by coniveau and by definition this filtration N^p satisfies

$$N^pH^k(X,\mathbb{Z}) \subseteq H^k(X,\mathbb{Z}) \cap H^{k-p,p}(X) \oplus \ldots \oplus H^{p,k-p}(X)$$

Hodge then claimed that we actually have an equality

Conjecture (The Generalized Hodge Conjecture)

$$N^pH^k(X,\mathbb{Z}) = H^k(X,\mathbb{Z}) \cap H^{k-p,p}(X) \oplus \ldots \oplus H^{p,k-p}(X)$$

As stated above, this conjecture is also false as proved by A. Grothendieck *(the other gentleman in the cover)*. Grothendieck argued that 'Hodge's general conjecture is false for trivial reasons' simply because the space on the right in the above equality need not to be a Hodge structure, he gave an example of a product of elliptic curves to illustrate the fact and offered a more convincing statement:

Conjecture (Grothendieck's Amended Hodge Conjecture[4]) $N^pH^k(X, \mathbb{Q})$ is the largest rational sub-Hodge structure of $H^{k-p,p}(X) \oplus \ldots \oplus H^{p,k-p}(X)$

This one remains to this day widely open.

[4]Grothendieck also made another conjecture related to this, the variational Hodge conjecture, in case one considers cycles spread in a family. Sadly we won't have time to discuss it today.

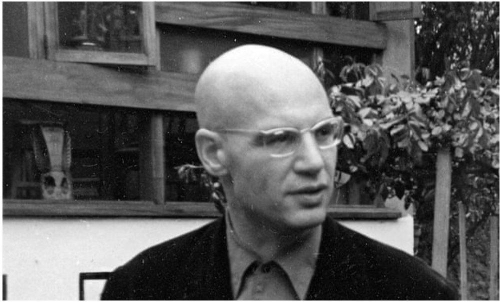


Figure: A man truly ahead of his time! Reformulated the entire field of algebraic geometry using the language of schemes. He said that *the Hodge conjecture is the deepest conjecture in the theory of complex algebraic varieties.*

Why not to generalize Lefschetz's proof?

Initial Attempts and known cases

After observing that the Hodge conjecture is just a generalization of a known theorem, an immediate question arrises: why can't we apply Lefschetz's arguments to prove the Hodge conjecture?

The short answer is: Jacobi Inversion fails, sometimes badly. In a few words, if the codimension is higher than 1 then the Abel-Jacobi map is not an isomorphism, even worse, sometimes its image is trivial. That fact prevent us to mimic Lefschetz's argument, in the sense that we can't build the algebraic cycle out of the associated normal function.

Not all hope is lost though...

Initial Attempts and known cases

Even though we can't generalize Lefschetz's proof, we can however analyze the case of varieties with many algebraic cycles (most of the time divisors) in it or varieties whose Chow/cohomology ring is well-known. For example:

- Projective spaces $\mathbb{P}^n$ and more generaly, Grassmanians $G(n,k)$(Cohomology generated by Schubert cycles which themselves are algebraic by construction).
- Hypersurfaces with many lines (or more technically when the cylinder homomorphism is surjective)
- Certain Fermat varieties $x_0^n + \ldots + x_n^n = 0$ in $\mathbb{P}^n$ (the conjecture is likely to be true for all of them)
- Numerous types of Abelian varieties, specially the ones whose cohomology ring is generated by divisors.

A surface with many lines

Initial Attempts and known cases

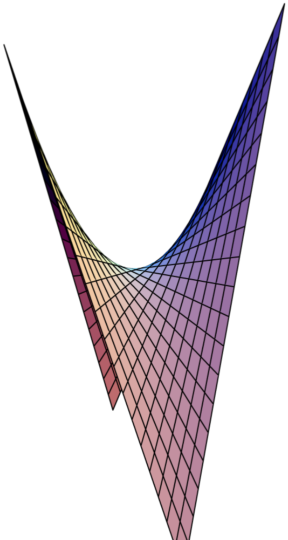


Figure: The quadric surface $xy - zw = 0$ in $\mathbb{P}^3$ has many lines on it, hence many algebraic cycles as well.

Around 2006, P. Griffiths and M. Green propose a new program to solve the Hodge conjecture. The approach is based on –like Lefschetz’s– normal functions. However, instead of looking at the algebraic cycle that induces the normal function, the idea is to look at the so called *singularities* of normal functions that can be detected (non torsion ones) only in codimension higher than one, which is exactly when Lefschetz’s approach fails.

The details of this program are very technical but it suffices to say that, to the best of my knowledge, no single major proof of the Hodge conjecture has surfaced after this approach was proposed.

Figure: Prof. Griffiths has always been an inspiration for me and was in part responsible for me getting a Ph.D. in the US!

Thank you!