THE HODGE-$\mathcal{D}$-CONJECTURE FOR A PRODUCT OF ELLIPTIC CURVES

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Abstract. Let $X/\mathbb{C}$ be a general product of elliptic curves. Our goal is to establish the Hodge-$\mathcal{D}$-conjecture for X. We accomplish this when $\dim X \leq 5$. For $\dim X \geq 6$, we reduce the conjecture to a matrix rank condition that is amenable to computer calculation.

In memory of B. Brent Gordon, July 7, 1953 - May 9, 2017.

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1. Introduction

Let $X/\mathbb{C}$ be a smooth projective variety of dimension n, and $\mathrm{CH}^k(X,m)$ the higher Chow group as defined in [B]. Our primary interest here are the K_1 classes $\mathrm{CH}^k(X,1)$. It is well known that a class $\xi \in \mathrm{CH}^k(X,1)$, can be represented in the form,

$$\xi = \sum_{i=1}^{N}(f_i, D_i),\ D_j \text{ irreducible},\ \mathrm{cd}_X D_i = k-1,\ f_i \in \mathbb{C}(D_i)^\times,\ \sum_{i=1}^{N}\mathrm{div}_{D_i}(f_i) = 0,$$

$\mathrm{CH}^k(X,1)$ being the group of all such cycles, modulo the image of the Tame symbol. If we were to drop the condition $\sum_{i=1}^{N}\mathrm{div}_{D_i}(f_i) = 0$, then ξ would be called a *precycle*. For any such precycle ξ, and real C^∞ test form ω of Hodge type $(n-k+1, n-k+1)$ on X, one has the integral calculation,

$$\sum_{i=1}^{N}\int_{D_i}\log|f_i|\omega \in \mathbb{R}.$$

Working on the level of cycle classes, there is the induced real regulator map,

$$\begin{aligned} r_{k,1} : \mathrm{CH}^k(X,1)\otimes_{\mathbb{Z}}\mathbb{R} &\to H^{k-1,k-1}(X,\mathbb{R}(k-1)) \\ &\simeq \left\{H^{n-k+1,n-k+1}(X,\mathbb{R}(n-k+1))\right\}^\vee, \end{aligned} \tag{1}$$

$$\omega \in H^{n-k+1,n-k+1}(X,\mathbb{R}(n-k+1)),\ r_{k,1}(\xi)(\omega) = \frac{1}{(2\pi\mathrm{i})^{n-k+1}}\sum_{i=1}^{N}\int_{D_i}\log|f_i|\omega.$$

In his attempt to put a rational structure on real Deligne cohomology, Beilinson once conjectured that $r_{k,1}$ is surjective[1], where $X/\mathbb{C}$ is viewed as a real variety via $X \to \mathrm{Sp}(\mathbb{C}) \to \mathrm{Sp}(\mathbb{R})$. That is now known to be false (see [MS], [C-L2], [C-L3]). Having said this, and to his credit, if $X/\mathbb{C}$ is obtained via base change from a smooth projective variety defined over a number field, then his conjecture for such X is probably true. On the other hand, there are interesting cases where the conjecture is true, such as for general Abelian and $K3$ surfaces [C-L1].

The goal of this paper is to prove the Hodge-$\mathcal{D}$-conjecture, viz., the surjectivity of $r_{k,1}$ in (1), for X a product of elliptic curves. Given the degeneration techniques of this paper, we first had to settle for general products of elliptic curves. The word "general" should be interpreted in the following sense. As the integral regulator to integral Deligne cohomology is holomorphic in a suitable sense, the real regulator is real analytic. There is the real analytic Zariski topology, where general refers to belonging to a real analytic open subset governed by certain generic properties. This should be compared to *very* general, which involves the complement of a *countable* union of proper real

[1]The conjecture is obviously true when $k = 1$.

analytic subvarieties. Secondly, we verify the Hodge-$\mathcal{D}$-conjecture for $n \le 4$; the situation $n \le 3$ also following from [C-L1]. One can argue that the case $n = 5$ already follows from the case $n \le 4$. For higher dimensions, the situation is much more amenable to numerical calculation. To this end, we set up everything for future computer computation.

Now for the road map of the paper.

1.) Our first step is to reduce to the case $\dim X = 2n$, viz., $X = E_1 \times \cdots \times E_{2n}$, $k = n+1$, and consider the map:

(2)
$$r_{n+1,1} : \mathrm{CH}^{n+1}(X,1) \otimes \mathbb{R} \to \big\{H^1(E_1,\mathbb{R}) \otimes \cdots \otimes H^1(E_{2n},\mathbb{R})(n)\big\} \bigcap H^{n,n}(X).$$

The anticipated surjectivity of $r_{n+1,1}$ in (2) will be referred to as the *primitive* Hodge-$\mathcal{D}$-conjecture.

2.) The right hand side of (2) has real dimension $\binom{2n}{n}$. Next, we come up with a list of candidate precycles, whose regulator values likely generate the right hand side of (2), for X general. This requires some explanation. In order to determine that a given precycle is nonzero for general X, we first degenerate to $\{y_j^2 = x_j^3;\ j = 1, ..., 2n\}$. This degeneration process turns out to be a two step process, carefully explained in [GL]. Then $\binom{2n}{n}$ precycle generators can be calculated by hand in the cases $n = 1, 2$. For $n > 2$, we need the aforementioned aid of a computer.

3.) Suppose we are given a precycle as in 2.), of the form say (f, D), where $\mathrm{cd}_X D = n$ and $f \in \mathbb{C}(D)^\times$, such that under degeneration as in 2.), the regulator value is nonzero. We construct another precycle (g, Z) say such that

(a) the regulator value of (g, Z) degenerates to zero,

(b) if we put $\xi := (f, D) - (g, Z)$, then $\mathrm{div}(\xi)$ is a linear combination of "horizontal and vertical cycles". The precise meaning of this will be explained in the body of this text, but the point is that ξ can be completed to a K_1 class $\overline{\xi} \in \mathrm{CH}^{n+1}(X,1)$, for which the regulator values of ξ and $\overline{\xi}$ coincide on test forms. The main results are given in Theorems 5.1 and 6.3.

We now draw the reader's attention to the earlier paper [GL]. One of the appealing aspects of [GL] is an explicit description of indecomposable K_1 classes in terms of defining equations, on a general surface $E_1 \times E_2$ and fourfold $E_1 \times E_2 \times E_3 \times E_4$. The problem is that in the case of a surface (and most likely the fourfold case), the K_1 class turned out instead to be *decomposable*[2], contrary to what was claimed in [GL]. The first author is grateful to M. Saito for pointing this out. This is explained in Addendum 4.1. In the case of a surface, there is a simple remedy. One simply chooses a precycle (f, D) on $X = E_1 \times E_2$, such that it's regulator value on test forms is nonzero, and for

[2]See Definition 6.4 for the meaning of (in)decomposable K_1 classes.

which $\mathrm{div}_D(f)$ involves the torsion points on E_j. The details are explained in [T]. This argument, however, does not extend to fourfolds - hence the current paper. Another point worth mentioning is that the precycles introduced in this paper for the surface and fourfold cases, are very similar to those in [GL]. As a consequence, the limit type arguments in [GL] extend to our situation. Having said this, there are new details in this paper warranted to deal with this new choice of precycles. Finally, [GL] was only focused on the nontriviality of $r_{n+1,1}$, for general surfaces and fourfolds. Our goals in this paper go much beyond this.

2. Notation

Much of the terminology has already been introduced in the introduction. The reader may wish to consult [GL] for additional information. Other than that, there is the Tate twist: Let $\mathbb{A} \subseteq \mathbb{R}$ be a subring. The Tate twist is given by $\mathbb{A}(r) = (2\pi \mathrm{i})^r\mathbb{A}$. It is the trivial $\mathbb{A}$-Hodge structure of weight $-2r$ and pure Hodge type $(-r, -r)$.

3. Reduction to a special case

Proposition 3.1. *Let $X = E_1 \times \cdots \times E_N$ be a product of elliptic curves, and consider the real regulator*

$$r_{k,1} : CH^k(X, 1) \otimes_{\mathbb{Z}} \mathbb{R} \to H^{k-1,k-1}(X, \mathbb{R}(k-1)).$$

Then $r_{k,1}$ is surjective iff for all $2 \leq \ell \leq k$, with $1 \leq i_1 < \cdots < i_{2\ell-2} \leq N$, $r_{\ell,1} : CH^\ell(E_{i_1} \times \cdots \times E_{i_{2\ell-2}}, 1)) \otimes \mathbb{R} \to \{H^1(E_{i_1}, \mathbb{R}) \otimes \cdots \otimes H^1(E_{i_{2\ell-2}}, \mathbb{R})(\ell - 1)\} \bigcap H^{\ell-1,\ell-1}(E_{i_1} \times \cdots \times E_{i_{2\ell-2}}, \mathbb{R}(\ell-1))$ is surjective.

Proof. The proof, which uses the functoriality of $r_{\bullet,1}$ and the Künneth formula, is left to the reader. It is also proven in [Ma]. $\square$

The import of Proposition 3.1 is that the Hodge-$\mathcal{D}$-conjecture for a general product of elliptic curves reduces to the primitive Hodge-$\mathcal{D}$-conjecture, viz., the surjectivity of $r_{n+1,1}$ in (2), for all $n \geq 1$.

The reader may find that the style of writing this paper may be somewhat cavalier - more like a stream of conciousness. There is good reason for this. First, the deformation details have already been worked out carefully in [GL]. Second, we know where to look for problems. See Addendum 4.1 below. Finally, additional details will also appear in [Ma]. The reader will be reminded of this periodically.

4. The surface case $X = E_1 \times E_2$

Write $y_j^2 = h_j(x_j)$, $h_j(x_j) = x_j^3 + b_j x_j + c_j$. Let $f_1 = x_1^2 x_2 + \mathrm{i}$, $f_2 = x_1^2 x_2 + 1$, $D = V(x_1x_2 + y_1y_2) \cap X$. Note that $\log|f_i| = \log 1 = 0$ on $V(x_1x_2)$. For general X, D is a smooth and irreducible curve [GL]. Let $p_\infty \in E_j$ be the point at infinity. When $c_j = 0$, we arrive at on D:

$$x_1x_2(x_1^2 + b_1)(x_2^2 + b_2) = h_1(x_1)h_2(x_2) = y_1^2y_2^2 = x_1^2x_2^2.$$

One then has

$$D = \{E_1 \times p_\infty + p_\infty \times E_2\} \ + \ D',$$

and that on D' we have

$$(x_1^2 + b_1)(x_2^2 + b_2) = x_1x_2 = -y_1y_2.$$

Further, if $b_j = 0$, then

$$D' = \{E_1 \times p_\infty + p_\infty \times E_2\} \ + \ D'',$$

where on D'',

$$x_1x_2 = 1, \quad y_1y_2 = -1.$$

Note that D'' is rational. Next, consider the holomorphic 1-form ω_j on smooth E_j, given by

$$\omega_j = \frac{dx_j}{y_j}.$$

Set

$$\omega_+ := \omega_1 \wedge \overline{\omega}_2 + \overline{\omega}_1 \wedge \omega_2, \quad \omega_- := \mathrm{i}\big(\omega_1 \wedge \overline{\omega}_2 - \overline{\omega}_1 \wedge \omega_2\big).$$

We consider

$$\int_D \log|f_i|\omega_\pm,$$

under degeneration to $y_j^2 = x_j^3$. This becomes:

$$\int_{D''} \log|f_i^\circ|\omega_\pm^\circ,$$

where on D'':

$$f_1^\circ = x_1 + \mathrm{i}, \ f_2^\circ = x_1 + 1, \ \omega_+^\circ = \frac{-4\mathrm{Im}(x_1)}{|x_1|^3}dA, \ \omega_-^\circ = \frac{4\mathrm{Re}(x_1)}{|x_1|^3}dA,$$

where $dA = d\mathrm{Re}(x_1) \wedge d\mathrm{Im}(x_1)$. Let $\mathcal{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}$ be the upper half plane in $\mathbb{C}$, and $\mathcal{H}_+ = \{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq 0\}$. And let's compute:

$$\int_{D''} \log|f_1^\circ|\omega_+^\circ = \int_{\mathbb{C}} \log|f_1^\circ|\omega_+^\circ = -4\int_{\mathbb{C}} \log|x_1 + \mathrm{i}|\frac{\mathrm{Im}(x_1)}{|x_1|^3}dA$$

$$= -4\int_{\mathcal{H}} \log\left|\frac{x_1 + \mathrm{i}}{\overline{x}_1 + \mathrm{i}}\right|\frac{\mathrm{Im}(x_1)}{|x_1|^3}dA < 0,$$

using

$$\left|\frac{x_1+\mathrm{i}}{\overline{x}_1+\mathrm{i}}\right| > 1 \Leftrightarrow \mathrm{Im}(x_1) > 0.$$

Note that for $z \in \mathbb{C}^\times$, $|z|/|\overline{z}| = 1$. Hence

$$\int_{D''} \log|f_2^\circ|\omega_+^\circ = -4\int_{\mathcal{H}} \log\left|\frac{x_1+1}{(\overline{x_1+1})}\right|\frac{\mathrm{Im}(x_1)}{|x_1|^3}dA = 0.$$

Similarly,

$$\int_{D''} \log|f_2^\circ|\omega_-^\circ = \int_{\mathbb{C}} \log|f_2^\circ|\omega_-^\circ = 4\int_{\mathbb{C}} \log|x_1+1|\frac{\mathrm{Re}(x_1)}{|x_1|^3}dA$$

$$= 4\int_{\mathcal{H}_+} \log\left|\frac{x_1+1}{-\overline{x}_1+1}\right|\frac{\mathrm{Re}(x_1)}{|x_1|^3}dA > 0,$$

using

$$\left|\frac{x_1+1}{-\overline{x}_1+1}\right| > 1 \Leftrightarrow \mathrm{Re}(x_1) > 0.$$

Finally,

$$\int_{D''} \log|f_1^\circ|\omega_-^\circ = 4\int_{\mathcal{H}_+} \log\left|\frac{x_1+\mathrm{i}}{-\overline{x}_1+\mathrm{i}}\right|\frac{\mathrm{Re}(x_1)}{|x_1|^3}dA = 0.$$

By degeneration arguments in [GL], we therefore deduce that for general X,

$$\det\begin{bmatrix}\int_D \log|f_1|\omega_+ & \int_D \log|f_1|\omega_- \\ \int_D \log|f_2|\omega_+ & \int_D \log|f_2|\omega_-\end{bmatrix} \neq 0. \tag{3}$$

The goal now is to extend (f_i, D) to K_1 classes on X, without compromising the nonzero calculation in (3). Let us work with say (f_1, D), and put $Z_1 = \overline{V(f_1)\cap X}$, $Z_2 = \overline{V(f_2)\cap X}$, $g_1 = x_1x_2 + y_1y_2 = g_2$. Clearly $(f_i, D) - (g_i, Z_i)$ is a step in the right direction. Let us show e.g. that

$$\int_{Z_1} \log|g_1|w_+ \mapsto 0, \text{ as } E_j \mapsto \{y_j^2 = x_j^3\}.$$

Under degeneration, we have $x_2 = -\mathrm{i}/x_1^2$, and $y_j^2 = x_j^3$. Taking differentials we have

$$dx_2 = \frac{2\mathrm{i}}{x_1^3}dx_1 = \frac{2\mathrm{i}}{y_1^2}dx_1.$$

Hence

$$\frac{dx_1}{y_1} \wedge \frac{d\overline{x}_2}{\overline{y}_2} = \frac{-2\mathrm{i}dx_1 \wedge d\overline{x}_1}{y_1\overline{y}_1^2\overline{y}_2} = \frac{-2\mathrm{i}y_1y_2}{|y_1|^4|y_2|^2}dx_1 \wedge d\overline{x}_1$$

Then

$$\omega_+^\circ = \frac{-2\mathrm{i}\mathrm{Re}(y_1y_2)}{|y_1|^4|y_2|^2}dx_1 \wedge d\overline{x}_1 = \frac{4\mathrm{Re}(y_1y_2)}{|y_1|^4|y_2|^2}dA = 4\mathrm{Re}(y_1y_2)dA.$$

Consider parameterizations for E_j:

$$(x_1, y_1) = (t^2, t^3), \quad (x_2, y_2) = (u^2, u^3).$$

Then

$$-\mathrm{i} = x_1^2 x_2 = t^4 u^2 \Rightarrow t^2 u = \pm\sqrt{-\mathrm{i}},\ tu = \pm\sqrt{-\mathrm{i}}/t, \&\ u = \pm\sqrt{-\mathrm{i}}/t^2,$$

and therefore

$$y_1 y_2 = t^3 u^3 = \frac{-\mathrm{i}u}{t} = \frac{\pm\mathrm{i}\sqrt{-\mathrm{i}}}{t^3} = \frac{\pm\mathrm{i}\sqrt{-\mathrm{i}}\ \bar{t}^3}{|t|^6} = \pm\bar{\mathbf{t}}^3/|\mathbf{t}|^6$$

where $\mathbf{t} = \overline{\sqrt{-\mathrm{i}}}t$. Next,

$$|g_1| = |x_1x_2 + y_1y_2| = |t^2u^2 + t^3u^3| = |t^2u^2||1 + tu| = \frac{1}{|\mathbf{t}|^2}\left|1 \pm \frac{\sqrt{-\mathrm{i}}}{t}\right|$$

$$= \frac{1}{|\mathbf{t}|^3}|t \pm \sqrt{-\mathrm{i}}| = \frac{1}{|\mathbf{t}|^3}|1 \pm \mathbf{t}|$$

Now write $\mathbf{t} = re^{\mathrm{i}\theta}$. Then

$$|1 \pm \mathbf{t}|^2 = (1 + r^2) \pm 2r\cos\theta = (1 + r^2)\left(1 \pm \frac{2r}{r^2 + 1}\cos\theta\right).$$

Calculating ω_+° is easy:

$$\omega_+^\circ = \frac{\pm 4\mathrm{Re}(\mathbf{t}^3)}{r^6} r dr d\theta = \frac{\pm 4\cos(3\theta)}{r^2} dr d\theta.$$

Note that by periodicity,

$$\int_0^{2\pi} \frac{\pm 4\cos(3\theta)}{r^2} d\theta = 0,$$

and therefore,

$$\int_0^{2\pi} \log\left[(1 + r^2)\left(1 \pm \frac{2r}{r^2 + 1}\cos\theta\right)\right] \frac{\pm 4\cos(3\theta)}{r^2} d\theta$$

$$= \int_0^{2\pi} \log\left(1 \pm \frac{2r}{r^2 + 1}\cos\theta\right) \frac{\pm 4\cos(3\theta)}{r^2} d\theta$$

$$= \pm 4\left[\int_0^{2\pi} \log\left(1 + \frac{2r}{r^2 + 1}\cos\theta\right) \frac{\cos(3\theta)}{r^2} d\theta\right.$$

$$\left.+ \int_0^{2\pi} \log\left(1 - \frac{2r}{r^2 + 1}\cos\theta\right) \frac{\cos(3\theta)}{r^2} d\theta\right].$$

Now taking Taylor series of the log function[3], this all amounts to the calculation of

$$(1+(-1)^m)\int_0^{2\pi}\cos(3\theta)\cos^m(\theta)d\theta, \tag{4}$$

for $m \geq 1$. The latter integral amounts to the third Fourier coefficient of $\cos^m\theta$. By a residue calculation[4], all such coefficients are zero for m even, and for m odd, the expression in (4) vanishes. Finally, by Fubini's theorem, it follows that

$$\int_{Z_1}\log|g_1|w_+ \mapsto 0, \text{ as } E_j \mapsto \{y_j^2 = x_j^3\}.$$

Likewise, it is also the case that

$$\int_{Z_1}\log|g_1|w_- \mapsto 0, \text{ as } E_j \mapsto \{y_j^2 = x_j^3\},$$

and that

$$\int_{Z_2}\log|g_2|w_\pm \mapsto 0, \text{ as } E_j \mapsto \{y_j^2 = x_j^3\}.$$

Our next step to to complete the precycles $\xi_i := (f_i, D) - (g_i, Z_i)$ to K_1 classes $\overline{\xi}_i$ on a general such X. Let's complete say ξ_1. Towards this goal, we consider homogeneous coordinates $([z_0, z_1, z_2], [w_0, w_1, w_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ (Segre), with $(x_1, y_1) = (z_1/z_0, z_2/z_0)$ and $(x_2, y_2) = (w_1/w_0, w_2/w_0)$. Then, in Segre coordinates,

$$D = V(z_1w_1 + z_2w_2) \cap X,\ f_1 = \frac{z_1^2w_1w_0 + \mathrm{i}z_0^2w_0^2}{z_0^2w_0^2},$$
$$Z_1 \ \subset V(z_1^2w_1w_0 + \mathrm{i}z_0^2w_0^2) \cap X,\ g_1 = \frac{z_1w_1 + z_2w_2}{z_0w_0}.$$

Then

$$\mathrm{div}_D(f_1) = V(z_1^2w_1w_0 + \mathrm{i}z_0^2w_0^2) \cap D \ - \ V(z_0^2w_0^2) \cap D,$$
$$\mathrm{div}_{Z_1}(g_1) = V(z_1w_1 + z_2w_2) \cap Z_1 \ - \ V(z_0w_0) \cap Z_1.$$

Hence

$$\mathrm{div}_D(f_1) \ - \ \mathrm{div}_{Z_1}(g_1) = V(z_0w_0) \cap Z_1 \ - \ V(z_0^2w_0^2) \cap D$$
$$= V(z_0) \cap Z_1 \ + V(w_0) \cap Z_1 \ - \ 2V(z_0) \cap D \ - \ 2V(w_0) \cap D$$

The situation is now very similar to the situation in §2.3 of [GL], to complete $\mathrm{div}_D(f_1) - \mathrm{div}_{Z_1}(g_1)$ to a K_1 class $\overline{\xi}_1$ on X, using horizontal and vertical curves of the form $E_1 \times \{q\}$, $\{p\} \times E_2$, and nonzero rational functions on E_1, E_2

[3]And observing that $\frac{2r}{r^2+1} < 1$ for $r \neq 1$.

[4]$\int_0^{2\pi}\cos(3\theta)\cos^m(\theta)d\theta = \int_{|z|=1}\left(\frac{z^3+z^{-3}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^m\frac{dz}{\mathrm{i}z}$. This amounts to the constant term of $\left(\frac{z^3+z^{-3}}{2}\right)\left(\frac{z+z^{-1}}{2}\right)^m$. But $\left(\frac{z+z^{-1}}{2}\right)^m = \sum_{j=0}^m\binom{m}{j}\frac{1}{2^m}z^jz^{-(m-j)}$. So we must solve $2j - m \pm 3 = 0 \Rightarrow m = 2j \pm 3$ is odd.

respectively. The details are left to the reader. They also will appear in [Ma]. Note that $H^1(E_1) \otimes H^1(E_2)$ pullsback to zero on these horizontal and vertical curves.

Corollary 4.1. *The Hodge-$\mathcal{D}$-conjecture holds for general X.*

Remark 4.2. *Corollary 4.1 also follows from theorem 8.1 in* [C-L1].

4.1. **Addendum: Where things can go wrong.** We point out the error in [GL]. The precycle in question regarding the surface $X = E_1 \times E_2$ was given by $(x_1 + \mathrm{i}, D)$. There are two degeneration arguments in §2 (op. cit.), the second of which involves a product $E_1 \times E_2$, where $E_j :\ y_j^2 = x_j^3 + b_j x_j$. If $t = (b_1, b_2)$, then our corresponding D_t is given by $E_1 \times E_2 \cap V(x_1x_2 + y_1y_2)$. This led to a description of D_t on $E_1 \times E_2$ involving (page 554, op. cit),

$$x_1^2x_2^2 = y_1^2y_2^2 = x_1x_2(x_1^2 + b_1)(x_2^2 + b_2).$$

Naturally, we could bleed off a factor x_1x_2 (i.e. a curve $E_1 \times (0,0) + (0,0) \times E_2$), because for smooth E_j, our real 2-form ω pulls back to zero on this. What's left however is the second family

$$\Sigma := \bigcup_{t \in U} D_t',$$

where D_t' is defined by the equation

$$x_1x_2 = (x_1^2 + b_1)(x_2^2 + b_2).$$

But when $t = 0$ (i.e. $(b_1, b_2) = (0, 0)$), we arrive at $x_1x_2 = x_1^2x_2^2$, hence

$$D_0' = D'' + \underbrace{E \times (0,0) + (0,0) \times E}_{\text{multiplicity } 1},$$

i.e. we must consider the case $x_1x_2 = 0$ as well. But since at $t = 0$, the real 2-form ω acquires singularities, it is no longer the case that we can ignore the factor $E \times (0,0) + (0,0) \times E$. In fact, the integral of $\log|x_1 + \mathrm{i}|\omega$ over this factor $E \times (0,0) + (0,0) \times E$ is the negative of the value of the integral of $\log|x_1 + \mathrm{i}|\omega$ over D''. Further, a standard a localization argument (suggested by M. Saito, given below) shows that our precycle $(x_1 + \mathrm{i}, D)$ completes to a decomposable cycle. There appears to be a natural remedy to this. Instead of using $\log|x_1 + \mathrm{i}|$, try $\log|x_1^2x_2 + \mathrm{i}|$. Then regarding the equation

$$x_1^2x_2^2 = x_1x_2,$$

it is clear that

$$\log|x_1^2x_2 + \mathrm{i}| = \begin{cases} \log|x_1 + \mathrm{i}| & \text{if } x_1x_2 = 1 \\ \log 1 = 0 & \text{if } x_1x_2 = 0 \end{cases}.$$

This issue now is to extend (f, D) to a K_1 cycle, where $f = x_1^2x_2+\mathrm{i}$, and a similar situation for the fourfold case, with $\log|x_1x_3-1|$ replaced by $\log|x_1^2x_2x_3^2x_4-1|$. This, and much more, is the import of the present paper.

Saito's comments, which rely on a careful perusal of [GL], are inspired by a construction of Nori, and explained in a paper of Schoen. It is based on the following observation. Let $f : X \to S$ be a smooth proper morphism of smooth varieties over k such that $\dim S = 1$. Let $\xi \in \mathrm{CH}^p(X, 1)$. By the localization sequence, it is decomposable if its restriction ξ_K to the generic fiber X_K of f vanishes, where $K = k(S)$. If ξ_K is decomposable in $\mathrm{CH}^p(X_K, 1)$, and is of the form $\xi' \otimes g$ with $\xi' \in \mathrm{CH}^{p-1}(X)$ and $g \in K$, then a multiple of ξ' vanishes if it's pull-back to the geometric generic fiber $X_{\bar{K}}$ vanishes. This argument applies to the cycle construction in §2.3 of [GL], where f is the first projection and $\xi' = [D] - 3[E_1 \times \{\infty\}]$.

5. The fourfold case $X = E_1 \times E_2 \times E_3 \times E_4$

We start off with $E_j = V(y_j^2 - h_j(x_j))$, $h_j(x_j) = x_j^3 + b_jx_j + c_j$, and $D = D_1\times D_2$, $D_1 = V(x_1x_2+y_1y_2)\cap E_1\times E_2$, $D_2 = V(x_3x_4+y_3y_4)\cap E_3\times E_4$. We first set $f_1 = x_1^2x_2x_3^2x_4 - 1$. Observe that $\log|f_1| = 0$ whenever $x_1x_2x_3x_4 = 0$. One has the forms $\omega_{1,\pm}$, $\omega_{2,\pm}$ living on D_1, D_2 respectively. Set $\omega_{\pm,\pm} = \omega_{1,\pm} \wedge \omega_{2,\pm}$ on D. Likewise one has $D'' := D_1'' \times D_2''$ where $x_1x_2 = 1 = -y_1y_2$ on D_1'', and $x_3x_4 = 1 = -y_3y_4$ on D_2''. Correspondingly, at $\{y_j^2 = x_j^3\}$, one has $\omega^\circ_{\pm,\pm} = \omega^\circ_{1,\pm} \wedge \omega^\circ_{2,\pm}$ on D''. Using the degeneration techniques in [GL], one shows that

$$\int_D \log|f_1|\omega_{\pm,\pm} \mapsto \int_{D''} \log|f_1^\circ|\omega^\circ_{\pm,\pm},$$

where $f_1^\circ = x_1x_3 - 1$ on D''. And so we first calculate,

$$\int_{D''} \log|f_1^\circ|\omega^\circ_{+,+},$$

which amounts to the calculation,

$$\int_{\mathbb{C}^2} \log|x_1x_3 - 1|\frac{\mathrm{Im}(x_1)\mathrm{Im}(x_3)}{|x_1x_2|^3}dV, \quad (dV = dx_1 \wedge d\overline{x}_1 \wedge dx_3 \wedge d\overline{x}_3),$$

$$= \int_{\mathbb{C}\times\mathcal{H}} \log\left|\frac{x_1x_3 - 1}{\overline{x}_1x_3 - 1}\right|\frac{\mathrm{Im}(x_1)\mathrm{Im}(x_3)}{|x_1x_2|^3}dV$$

$$= \int_{\mathcal{H}^2} \log\left|\frac{(x_1x_3 - 1)(\overline{x}_1\overline{x}_3 - 1)}{(\overline{x}_1x_3 - 1)(x_1\overline{x}_3 - 1)}\right|\frac{\mathrm{Im}(x_1)\mathrm{Im}(x_3)}{|x_1x_2|^3}dV \ > \ 0,$$

since

$$\log\left|\frac{(x_1x_3 - 1)(\overline{x}_1\overline{x}_3 - 1)}{(\overline{x}_1x_3 - 1)(x_1\overline{x}_3 - 1)}\right| > 0 \Leftrightarrow \left|\frac{(x_1x_3 - 1)(\overline{x}_1\overline{x}_3 - 1)}{(\overline{x}_1x_3 - 1)(x_1\overline{x}_3 - 1)}\right|^2 > 1$$

$$\Leftrightarrow \mathrm{Im}(x_1)\mathrm{Im}(x_3) > 0.$$

Our next step is to extend the precycle $\xi_1 := (f_1, D)$ to a K_1 class $\overline{\xi}_1$ on X. As in the surface case, and towards this goal, we put $g_1 = x_1x_2 + y_1y_2$, and set

$$Z_1 = \overline{V(x_1^2x_2x_3^2x_4 - 1, x_3x_4 + y_3y_4) \cap X}.$$

We want to show that

$$\int_{Z_1} \log|g_1|\omega_{+,+} \mapsto 0, \text{ as } E_j \mapsto \{y_j^2 = x_j^3\}.$$

Recall that $\omega_{+,+} = \omega_{1,+} \wedge \omega_{2,+}$, and by the definition of Z_1,

$$\omega_{2,+} \mapsto \omega_{2,+}^\circ = \frac{-4\mathrm{Im}(x_3)}{|x_3|^3}dA_2, \quad \text{where } dA_2 = (\mathrm{i}/2)dx_3 \wedge d\overline{x}_3.$$

For $\omega_{1,+}$ we need more relations from Z_1. Namely $x_1^2x_2x_3^2x_4 = 1$, with in the limit, $x_3x_4 = 1 = -y_3y_4$. Hence $x_1^2x_2x_3 = 1$. Taking differentials, we arrive at:

$$2x_1x_2x_3dx_1 + x_1^2x_3dx_2 \equiv 0 \text{ mod } dx_3,$$

hence

$$2x_2dx_1 + x_1dx_2 \equiv 0 \text{ mod } dx_3, \Rightarrow dx_2 \equiv (-2x_2/x_1)dx_1 = \frac{-2}{x_1^3x_3}dx_1.$$

Now recall

$$\omega_{1,+} = \frac{dx_1}{y_1} \wedge \frac{d\overline{x}_2}{\overline{y}_2} + \frac{d\overline{x}_1}{\overline{y}_1} \wedge \frac{dx_2}{y_2},$$

and modulo $dx_3,\ d\overline{x}_3$,

$$\equiv \mathrm{Im}(\Lambda)dA_1, \quad (dA_1 = \frac{\mathrm{i}}{2}dx_1 \wedge d\overline{x}_1),$$

where,

$$\Lambda = \frac{-4y_1y_2x_3}{|y_1|^4|y_2|^2|x_3|^2}. \text{ Set } \Lambda_0 = -\Lambda/4 = \frac{y_1y_2x_3}{|y_1|^4|y_2|^2|x_3|^2}.$$

The calculation of

$$\int_{Z_1} \log|g_1|\omega_{+,+} \text{ as } E_j \mapsto \{y_j^2 = x_j^3\},$$

amounts to a Fubini integral calculation of the form:

$$\int_{\mathbb{C}}\left(\int_{\mathbb{C}} \log|x_1x_2 + y_1y_2|^2\mathrm{Im}(\Lambda_0)dA_1\right)\mathrm{Im}\left(\frac{x_3}{|x_3|^3}\right)dA_2.$$

We now consider parameterizations for $\{y_j^2 = x_j^3\}$:

$$(x_1, y_1) = (t^2, t^3), \quad (x_2, y_2) = (u^2, u^3), \quad (x_3, y_3) = (v^2, v^3).$$

(No need for parametrizing (x_4, y_4).) Now with respect to these parameterizations,

$$\Lambda_0 = \frac{t^3u^3v^2}{|t|^{12}|u|^6|v|^4},\ 1 = x_1^2x_2x_3 = t^4u^2v^2,\ \Rightarrow u = \frac{\pm 1}{t^2v},$$

$$x_1x_2 + y_1y_2 = t^2u^2 + t^3u^3 = t^2u^2(1+tu) = \frac{1 \pm 1/(tv)}{t^2v^2} = \frac{tv \pm 1}{t^3v^3}.$$

We also have

$$\Lambda_0 \equiv \pm \bar{t}^3 \bar{v} \quad \text{modulo powers of } |t|,\ |v|,$$

and correspondingly,

$$|x_1x_2 + y_1y_2|^2 \equiv |1 \pm tv|^2.$$

Now consider polar coordinates, viz., $t = r_1 e^{i\theta_1}$ and $v = r_2 e^{i\theta_2}$, $r_1 = |t|$ and $r_2 = |v|$. Then $dA_1 = r_1 dr_1 d\theta_1$, $dA_2 = r_2 dr_2 d\theta_2$. Our preoccupation will concern integrating with respect to θ_1. Then:

$$\Lambda_0 \equiv \pm e^{-i(3\theta_1 + \theta_2)},\ \Rightarrow \mathrm{Im}(\Lambda_0) \equiv \pm \sin(3\theta_1 + \theta_2).$$

Further,

$$|x_1x_2 + y_1y_2|^2 \equiv |1 \pm r_1r_2 e^{i(\theta_1+\theta_2)}|^2 = [1 \pm r_1r_2\cos(\theta_1+\theta_2)]^2 + r_1^2r_2^2\sin^2(\theta_1+\theta_2).$$

$$= (1 + r_1^2r_2^2) \pm 2r_1r_2\cos(\theta_1+\theta_2) \equiv 1 \pm \frac{2r_1r_2\cos(\theta_1+\theta_2)}{1+r_1^2r_2^2}.$$

And accordingly,

$$\log|x_1x_2 + y_1y_2|^2 \equiv \log\left(1 \pm \frac{2r_1r_2\cos(\theta_1+\theta_2)}{1+r_1^2r_2^2}\right). \tag{5}$$

By applying a Taylor series description of the RHS of (5), the calculation of

$$\int_{Z_1} \log|g_1|\omega_{+,+} \text{ as } E_j \mapsto \{y_j^2 = x_j^3\},$$

rests on the calculation of

$$[1 + (-1)^m]\int_0^{2\pi} \cos^m(\theta_1+\theta_2)\sin(3\theta_1+\theta_2)d\theta_1 \tag{6}$$

Now using

$$\sin(3\theta_1 + \theta_2) = \sin(3[\theta_1+\theta_2] - 2\theta_2)$$

$$= \sin(3[\theta_1+\theta_2])\cos(2\theta_2) - \cos(3[\theta_1+\theta_2])\sin(2\theta_2),$$

together with $\cos/\sin$ = even/odd, periodicity, plus arguments in the surface case, we deduce that the integral in (6) is zero. Therefore we conclude that

$$\int_{Z_1} \log|g_1|\omega_{+,+} \mapsto 0,\ \text{ as } E_j \mapsto \{y_j^2 = x_j^3\}.$$

5.1. **A K_1 class on X.** We wish to extend the precycle (f_1, D) on a general X, to a K_1 class on X. As a first step in this direction, we consider completing $\xi_1 := (f_1, D) - (g_1, Z_1)$ to a K_1 class $\overline{\xi}_1$. To analyze this further, we consider projective coordinates:

$$([t_0, t_1, t_2], [u_0, u_1, u_2], [v_0, v_1, v_2], [w_0, w_1, w_2]) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2.$$

with corresponding affine coordinates:

$$(x_1, y_1) = (t_1/t_0, t_2/t_0), \quad (x_2, y_2) = (u_1/u_0, u_2/u_0),$$

$$(x_3, y_3) = (v_1/v_0, v_2/v_0), \quad (x_4, y_4) = (w_1/w_0, w_2/w_0).$$

Then D is given by $V(t_1u_1 + t_2u_2, v_1w_1 + v_2w_2) \cap X$. Likewise, $Z_1 = \overline{V(g_1) \cap \{E_1 \times E_2 \times D_2\}}$. Furthermore, in Segre coordinates,

$$f_1 = \frac{t_1^2u_1u_0v_1^2w_1w_0 - t_0^2u_0^2v_0^2w_0^2}{t_0^2u_0^2v_0^2w_0^2}, \quad g_1 = \frac{t_1u_1t_0u_0 + t_2u_2t_0u_0}{t_0^2u_0^2}.$$

Then $\mathrm{div}(\xi_1)$ is supported on

$$V(t_0)\cap Z_1 + V(u_0)\cap Z_1 + V(t_0)\cap D + V(u_0)\cap D + V(v_0)\cap D + V(w_0)\cap D.$$

We refer to these as "horizontal and vertical cycles". This situation now of extending ξ_1 to $\overline{\xi}_1$ is very similar to that on page 562 of [GL], and will be left to the reader. This will also appear in [Ma].

5.2. **Hodge-$\mathcal{D}$-conjecture for general such X.** We have now constructed a K_1 class $\overline{\xi}_1$ on X, by showing that $r_{3,1}(\overline{\xi}_1)(\omega_{+,+}) \neq 0$. This was accomplished by first showing that

$$\int_{D''} \log|f_1^\circ|\omega_{+,+}^\circ \neq 0.$$

This amounted to the calculation

$$\int_{\mathcal{H}^2} \log\left|\frac{(x_1x_3 - 1)(\overline{x}_1\overline{x}_3 - 1)}{(\overline{x}_1x_3 - 1)(x_1\overline{x}_3 - 1)}\right| \frac{\mathrm{Im}(x_1)\mathrm{Im}(x_3)}{|x_1x_3|^3} dV \ > 0,$$

where we observed that

$$\left|\frac{(x_1x_3 - 1)(\overline{x}_1\overline{x}_3 - 1)}{(\overline{x}_1x_3 - 1)(x_1\overline{x}_3 - 1)}\right| > 1 \Leftrightarrow \mathrm{Im}(x_1)\mathrm{Im}(x_3) > 0.$$

To accomplish our goals in this section, it will first be necessary to calculate

$$\int_{D''} \log|f_1^\circ|\omega_{-,+}^\circ, \ \int_{D''} \log|f_1^\circ|\omega_{+,-}^\circ, \ \int_{D''} \log|f_1^\circ|\omega_{-,-}^\circ.$$

One can the easily show that, regarding

$$\int_{D''} \log|f_1^\circ|\omega_{-,+}^\circ,$$

this is reduced to calculating

$$\int_{\mathcal{H}\times\mathcal{H}_+} \log\left|\frac{(x_1x_3-1)(\overline{x}_1\overline{x}_3+1)}{(\overline{x}_1x_3+1)(x_1\overline{x}_3-1)}\right| \frac{\mathrm{Re}(x_1)\mathrm{Im}(x_3)}{|x_1x_3|^3}dV \ = 0, \tag{7}$$

for the following reason. One shows that

$$\left|\frac{(x_1x_3-1)(\overline{x}_1\overline{x}_3+1)}{(\overline{x}_1x_3+1)(x_1\overline{x}_3-1)}\right| > 1 \Leftrightarrow \mathrm{Im}(x_1)\mathrm{Im}(x_3)\mathrm{Re}(x_1)\mathrm{Re}(x_3) > 0 \text{ on } \mathbb{C}^2$$

$$\Leftrightarrow \mathrm{Re}(x_1)\mathrm{Im}(x_3) > 0 \text{ on } \mathcal{H}\times\mathcal{H}_+.$$

One then writes $\mathcal{H}\times\mathcal{H}_+ = \Omega_+\cup\Omega_-$ in terms of $\Omega_+ : \mathrm{Re}(x_1)\mathrm{Im}(x_3) \geq 0$ and $\Omega_- : \mathrm{Re}(x_1)\mathrm{Im}(x_3) \leq 0$. The integral in (7) can be broken into a sum of integrals over Ω_+ and Ω_-. Applying a change of variables formula, the sum of these integrals is zero. Similar story regarding

$$\int_{D''} \log|f_1^\circ|\omega^\circ_{+,-},$$

i.e., one is led to a zero limit calculation[5] . As for

$$\int_{D''} \log|f_1^\circ|\omega^\circ_{-,-},$$

this reduces to the calculation

$$\int_{\mathcal{H}_+^2} \log\left|\frac{(x_1x_3-1)(\overline{x}_1\overline{x}_3-1)}{(-\overline{x}_1x_3-1)(-x_1\overline{x}_3-1)}\right| \frac{\mathrm{Re}(x_1)\mathrm{Re}(x_3)}{|x_1x_2|^3}dV \ < 0,$$

since

$$\left|\frac{(x_1x_3-1)(\overline{x}_1\overline{x}_3-1)}{(-\overline{x}_1x_3-1)(-x_1\overline{x}_3-1)}\right| < 1 \Leftrightarrow \mathrm{Re}(x_1)\mathrm{Re}(x_3) > 0.$$

Recall that

$$\omega_j = \frac{dx_j}{y_j},$$

and now set

$$\begin{aligned}
\eta_{1,+} &= \omega_1\wedge\omega_2\wedge\overline{\omega}_3\wedge\overline{\omega}_4 + \overline{\omega}_1\wedge\overline{\omega}_2\wedge\omega_3\wedge\omega_4\\
\eta_{1,-} &= \mathrm{i}\big(\omega_1\wedge\omega_2\wedge\overline{\omega}_3\wedge\overline{\omega}_4 - \overline{\omega}_1\wedge\overline{\omega}_2\wedge\omega_3\wedge\omega_4\big)\\
\eta_{2,+} &= \omega_1\wedge\omega_3\wedge\overline{\omega}_2\wedge\overline{\omega}_4 + \overline{\omega}_1\wedge\overline{\omega}_3\wedge\omega_2\wedge\omega_4\\
\eta_{2,-} &= \mathrm{i}\big(\omega_1\wedge\omega_3\wedge\overline{\omega}_2\wedge\overline{\omega}_4 - \overline{\omega}_1\wedge\overline{\omega}_3\wedge\omega_2\wedge\omega_4\big)\\
\eta_{3,+} &= \omega_1\wedge\omega_4\wedge\overline{\omega}_2\wedge\overline{\omega}_3 + \overline{\omega}_1\wedge\overline{\omega}_4\wedge\omega_2\wedge\omega_3\\
\eta_{3,-} &= \mathrm{i}\big(\omega_1\wedge\omega_4\wedge\overline{\omega}_2\wedge\overline{\omega}_3 - \overline{\omega}_1\wedge\overline{\omega}_4\wedge\omega_2\wedge\omega_3\big)
\end{aligned}$$

[5]The vanishing of these two integrals can also be easily established using the techniques in §6.

Notice that $\eta_{1,\pm}\big|_{D''} = 0$. Note that

$$\dim_{\mathbb{C}} \left\{H^1(E_1,\mathbb{C})\otimes\cdots\otimes H^1(E_4,\mathbb{C})\right\}\cap H^{2,2}(X) = \binom{4}{2} = 6,$$

and that a real basis for

$$\left\{H^1(E_1,\mathbb{R})\otimes\cdots\otimes H^1(E_4,\mathbb{R})\right\}\cap H^{2,2}(X),$$

is given by either $\{\omega_{\pm,\pm},\eta_{1,\pm}\}$ or $\{\eta_{j,\pm}, j=1,2,3\}$. We seek to express $\{\eta_{j,\pm}\}$ in terms of $\omega_{\pm,\pm}$ and $\eta_{1,\pm}$. An explicit calculation gives:

$$\omega_{+,+} = -\eta_{2,+} + \eta_{3,+} \quad ; \quad \omega_{-,-} = \eta_{2,+} + \eta_{3,+}$$
$$\omega_{+,-} = -\eta_{2,-} - \eta_{3,-} \quad ; \quad \omega_{-,+} = -\eta_{2,-} + \eta_{3,-}$$

Correspondingly,

$$\eta_{2,+} = \frac{\omega_{-,-}-\omega_{+,+}}{2} \qquad \eta_{2,-} = -\left[\frac{\omega_{+,-}+\omega_{-,+}}{2}\right]$$
$$\eta_{3,+} = \frac{\omega_{+,+}+\omega_{-,-}}{2} \qquad \eta_{3,-} = \frac{\omega_{-,+}-\omega_{+,-}}{2}$$

First, we observe that after degeneration to $\{y_j^2 = x_j^3\}$, and up to twist,

$$\big(r_{3,1}(\overline{\xi}_1)(\eta_{1,+}), r_{3,1}(\overline{\xi}_1)(\eta_{1,-}), r_{3,1}(\overline{\xi}_1)(\omega_{+,+}), r_{3,1}(\overline{\xi}_1)(\omega_{-,-}), r_{3,1}(\overline{\xi}_1)(\omega_{+,-}),$$
$$r_{3,1}(\overline{\xi}_1)(\omega_{-,+})\big) \mapsto (0,0,[+],[-],0,0), \tag{8}$$

where $[+] > 0$ and $[-] < 0$. Furthermore, the reader can easily check, using the change of variables formula, that $[+]+[-1] = 0$. Then again after degeneration to $\{y_j^2 = x_j^3\}$, and for some $* \neq 0 \in \mathbb{R}$,

$$\big(r_{3,1}(\overline{\xi}_1)(\eta_{1,+}), r_{3,1}(\overline{\xi}_1)(\eta_{1,-}), r_{3,1}(\overline{\xi}_1)(\eta_{2,+}), r_{3,1}(\overline{\xi}_1)(\eta_{2,-}), r_{3,1}(\overline{\xi}_1)(\eta_{3,+}),$$
$$r_{3,1}(\overline{\xi}_1)(\eta_{3,-})\big) \mapsto (0,0,*,0,0,0). \tag{9}$$

Now let's recall that the K_1-class $\overline{\xi}_1$ originated from the precycle (f_1, D), where $D = D_1\times D_2$, $D_1 \subset E_1\times E_2$, $D_2\subset E_3\times E_4$, and $f_1 = x_1^2x_2x_3^2x_4 - 1$. Now with the same D, let $f_2 = x_1^2x_2x_3^2x_4 - \mathrm{i}$ [6]. Then the precycle (f_2, D) likewise extends to a K_1 class $\overline{\xi}_2$ on X. The corresponding values to that of (8), and of (9) with $[\pm]$, $* \neq 0$ in different slots. Next, consider the same type of arrangement $\underline{D} = \underline{D}_1 \times \underline{D}_2$, where $\underline{D}_1 \subset E_1\times E_3$ and $\underline{D}_2 \subset E_2\times E_4$. We then consider K_1 classes $\overline{\xi}_3$, $\overline{\xi}_4$ arising from the precycles $(f_3,\underline{D})$, $(f_4,\underline{D})$ respectively, where $f_3 = x_1^2x_3x_2^2x_4 - 1$ and $f_4 = x_1^2x_3x_2^2x_4 - \mathrm{i}$. Finally, consider $\underline{\underline{D}} = \underline{\underline{D}}_1 \times \underline{\underline{D}}_2$, where $\underline{\underline{D}}_1 \subset E_2\times E_3$, $\underline{\underline{D}}_2 \subset E_1\times E_4$, and with precycles $(f_5,\underline{\underline{D}})$, $(f_6,\underline{\underline{D}})$ with $f_5 = x_2^2x_3x_1^2x_4 - 1$ and $f_6 = x_2^2x_3x_1^2x_4 - \mathrm{i}$, and corresponding K_1 classes $\overline{\xi}_5$, $\overline{\xi}_6$. The ambitious reader can check that the corresponding regulator values with respect to the basis $\{\eta_{j,\pm}\}$ limit to the same as in (9), with $*$ in a different

[6] Same story if instead we use $f = x_1^2x_2x_3^2x_4 + \mathrm{i}$. But let's not.

slot. In particular, regulator values with respect to the basis $\{\eta_{j,\pm}\}$ limit to a nonsingular 6×6 matrix. This is corroborated with the results in Appendix 7.

Theorem 5.1. *For general X, $\{r_{3,1}(\bar{\xi}_j);\ j=1,\dots,6\}$ is an independent set in $H^{2,2}(X,\mathbb{R}(2))$. Indeed the Hodge-$\mathcal{D}$-conjecture holds for X.*

6. The general case

We now consider the situation of general $X = E_1\times\cdots\times E_N$. According to Proposition 3.1, we may assume that $N=2n$ is even and show that the regulator,

(10)
$$r_{n+1,1} : \mathrm{CH}^{n+1}(X,1)\otimes\mathbb{R} \to \big\{H^1(E_1,\mathbb{R})\otimes\cdots\otimes H^1(E_{2n},\mathbb{R})(n+1)\big\}\bigcap H^{n,n}(X),$$

is surjective. One can show that the real dimension of the RHS of (10) is $\binom{2n}{n}$. To grease the skids, we'll start with a specific case situation. Let (x_j,y_j) be affine coordinates of E_j with $y_j^2 = h_j(x_j)$. We start off with $D = D_1\times\cdots\times D_n\subset X$, where $D_i = V(x_{2i-1}x_{2i}+y_{2i-1}y_{2i})\cap E_{2i-1}\times E_{2i}$, $i=1,\dots,n$. We put $f_1 = x_1^2x_2x_3^2x_4\cdots x_{2n-1}^2x_{2n}\ -\ 1$. Further, let $\omega = \omega_{1,+}\wedge\cdots\wedge\omega_{n,+}$. Now under degeneration to $y_j^2 = x_j^3$,

$$\int_D \log|f_1|\omega \mapsto \int_{D''}\log|f_1^\circ|\omega^\circ,$$

where $D'' = D_1''\times\cdots\times D_n''$, $x_{2i-1}x_{2i} = 1 = -y_{2i-1}y_{2i}$ on D_i'',

$$f_1^\circ = x_1x_3\cdots x_{2n-1}\ -\ 1,$$

and

$$\omega^\circ = \omega_{1,+}^\circ\wedge\cdots\wedge\omega_{n,+}^\circ = \frac{\mathrm{Im}(x_1)\mathrm{Im}(x_3)\cdots\mathrm{Im}(x_{2n-1})}{|x_1x_3\cdots x_{2n-1}|^3}dV,$$

where dV, up to multiplicative constant, is given by,

$$dV = dA_1\wedge\cdots\wedge dA_n,\quad dA_i = d\mathrm{Re}(x_{2i-1})\wedge d\mathrm{Im}(x_{2i-1}).$$

Now write $x_{2i-1} = r_i\mathrm{e}^{\mathrm{i}\theta_i}$. The integral $\int_{D''}\log|f_1^\circ|\omega^\circ$ amounts to a Fubini calculation of the form,

(11)
$$\int_0^\infty\cdots\int_0^\infty\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{\log\big|(r_1\cdots r_n)\mathrm{e}^{\mathrm{i}(\theta_1+\cdots+\theta_n)}-1\big|^2}{r_1\cdots r_n}\bigg(\prod_{i=1}^n\sin\theta_i\bigg)\begin{matrix}d\theta_1\cdots d\theta_n\\ dr_1\cdots dr_n\end{matrix}.$$

Note that

$$\begin{aligned}\big|(r_1\cdots r_n)\mathrm{e}^{\mathrm{i}(\theta_1+\cdots+\theta_n)}-1\big|^2 &= \big(1-(r_1\cdots r_n)\cos(\theta_1+\cdots+\theta_n)\big)^2\\ &\quad+(r_1\cdots r_n)^2\sin^2(\theta_1+\cdots+\theta_n).\\ &= \big(1+(r_1\cdots r_n)^2)\big)\ -\ 2r_1\cdots r_n\cos(\theta_1+\cdots+\theta_n).\end{aligned}$$

Observe that

$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\log\big(1+(r_1\cdots r_n)^2)\big)}{r_1\cdots r_n}\bigg(\prod_{i=1}^n \sin\theta_i\bigg)\frac{d\theta_1\cdots d\theta_n}{dr_1\cdots dr_n} = 0.$$

Thus, the integral in (11) takes the form,

$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\log\big(1-\frac{2(r_1\cdots r_n)\cos(\theta_1+\cdots+\theta_n)}{1+(r_1\cdots r_n)^2}\big)}{r_1\cdots r_n}\bigg(\prod_{i=1}^n \sin\theta_i\bigg)\frac{d\theta_1\cdots d\theta_n}{dr_1\cdots dr_n},$$

which is easily seen to be finite dimensional. For this reason we can forego integration with respect to $dr_1\cdots dr_n$, and accordingly study the nature of the integral,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \frac{\log\big(1-\frac{2(r_1\cdots r_n)\cos(\theta_1+\cdots+\theta_n)}{1+(r_1\cdots r_n)^2}\big)}{r_1\cdots r_n}\bigg(\prod_{i=1}^n \sin\theta_i\bigg)d\theta_1\cdots d\theta_n. \tag{12}$$

Taking the Taylor series of $\log(1+x)$, this amounts to studying,

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \cos^m(\theta_1+\cdots+\theta_n)\bigg(\prod_{i=1}^n \sin\theta_i\bigg)d\theta_1\cdots d\theta_n,\ m\geq 1. \tag{13}$$

When $n=1$, the integral in (13) is zero. This is the surface case, and it is entirely consistent with the results in §4. If we instead were to set $f = x_1^2x_2x_3^2x_4\cdots x_{2n-1}^2x_{2n} - \mathrm{i}$, then the situation in the case $n=1$ would be very different. Let's examine the case $n=2$, viz., X a fourfold. In this case (13) becomes,

$$\int_0^{2\pi}\int_0^{2\pi} \cos^m(\theta_1+\theta_2)\sin\theta_1\sin\theta_2 d\theta_1 d\theta_2. \tag{14}$$

Next, applying the binomial formula to

$$\cos^m(\theta_1+\theta_2) = \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\big)^m,$$

the integral in (14) is reduced to the calculation of

$$(-1)^\ell \int_0^{2\pi}\int_0^{2\pi} \big(\cos\theta_1\cos\theta_2\big)^{m-\ell}\big(\sin\theta_1\sin\theta_2\big)^{\ell+1}d\theta_1 d\theta_2, \quad 0\leq \ell\leq m,$$

which in turn is the same as

$$(-1)^\ell\bigg(\int_0^{2\pi} \cos^{m-\ell}\theta\sin^{\ell+1}\theta d\theta\bigg)^2. \tag{15}$$

If $\ell = 2k$ is even, then the integrand in (15) becomes

$$\cos^{m-\ell}\theta(1-\cos^2\theta)^k\sin\theta,$$

and the integral in (15) is then zero. We therefore need to show that there are odd ℓ and m, such that the integral in (15) is nonzero. That's easy; just choose $m = \ell = 1$. Consequently, it follows from this that in the case $n = 2$,

$$\int_{D''} \log |f_1^\circ| \omega^\circ \neq 0,$$

that we already showed in §5 by different means. Now suppose instead, we set

$$f_2 = x_1^2 x_2 \cdots x_{2n-1}^2 x_{2n} \ - \ \mathrm{i}.$$

Then the corresponding integral to that in (13) above becomes

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \sin^m(\theta_1 + \cdots + \theta_n) \Big(\prod_{i=1}^n \sin\theta_i \Big) d\theta_1 \cdots d\theta_n, \ m \geq 1. \tag{16}$$

We consider the following

Statement 6.1. *In each of the integrals in (13) and (16), the sign does not depend on $m \geq 1$. Moreover:*

(i) *Regarding (13), it vanishes when n is odd. For n even, there is an m for which it doesn't vanish.*

(ii) *Regarding (16), it vanishes when n is even. For n odd, there is an m for which it doesn't vanish.*

By a brute force calculation in the case $n = 3$, it easily follows that Statement 6.1 holds for $n = 1, 2, 3$. The details will appear in [Ma].

Proposition 6.2. *If Statement 6.1 holds for all $n \geq 1$, then the image of the regulator in (10) is nonzero.*

Proof. Consider precycles (f_1, D), (f_2, D) and put

$$Z_1 = \overline{V(f_1) \cap \{E_1 \times E_2\} \times D_2 \times \cdots \times D_n},$$

$$Z_2 = \overline{V(f_2) \cap \{E_1 \times E_2\} \times D_2 \times \cdots \times D_n}.$$

$$g_1 = g_2 = x_1 x_2 + y_1 y_2.$$

Then it is easy to show that $\xi_i := (f_i, D) - (g_i, Z_i)$ is a linear combination of horizontal and vertical cycles. One can then complete ξ_i to a K_1 class $\overline{\xi}_i$ on X, without affecting the integrals in question. Finally, one shows that

$$\int_{Z_i} \log |g_i| \omega \mapsto 0, \ \text{as} \ E_j \mapsto \{y_j^2 = x_j^3\},$$

involving a direct generalization of a similar argument in §5. The rest is clear. $\square$

6.1. **Hodge-$\mathcal{D}$-conjecture.** We return to the subject title of this paper. Let $X = E_1 \times \cdots \times E_{2n}$. It is clear that

$$\dim_{\mathbb{C}}\Big\{H^{n,n}(X)\bigcap\big\{H^1(E_1,\mathbb{C})\otimes\cdots\otimes H^1(E_{2n},\mathbb{C})\big\}\Big\} = \binom{2n}{n}.$$

It is also the case that

$$\dim_{\mathbb{R}}\Big\{H^{n,n}(X)\bigcap\big\{H^1(E_1,\mathbb{R})\otimes\cdots\otimes H^1(E_{2n},\mathbb{R})\big\}\Big\} = \binom{2n}{n}.$$

This will be made implicitly clear from what follows. Let

$$\Phi(k,n) = \{1 \leq i_1 < \cdots < i_k \leq n\} \subset \{1,...,n\}.$$

We are interested in $\Phi(n,2n)$, where we observe that $|\Phi(n,2n)| = \binom{2n}{n}$. For a given $I \in \Phi(n,2n)$, there is a unique $J \in \Phi(n,2n)$ such that $I \cup J = \{1,...,2n\}$. Recall the holomorphic 1-forms $\{\omega_1,...,\omega_{2n}\}$. Then $\omega_I \wedge \overline{\omega}_J$ is a complex (n,n) form on X. Let's put

$$\eta_+ = \omega_I \wedge \overline{\omega}_J + \overline{\omega}_I \wedge \omega_J,$$

$$\eta_- = \mathrm{i}\big(\omega_I \wedge \overline{\omega}_J - \overline{\omega}_I \wedge \omega_J\big).$$

Note that if we place $\omega_I \wedge \overline{\omega}_J$ by $\omega_J \wedge \overline{\omega}_I$, then up to sign, we get the same $\eta_\pm$. Translated, one can find $N = \frac{1}{2}\binom{2n}{n}$ complex (n,n) forms by this process, such that correspondingly, there is a real basis $\{\eta_{1,\pm},...,\eta_{N,\pm}\}$ of $H^{n,n}(X)\bigcap\big\{H^1(E_1,\mathbb{R})\otimes\cdots\otimes H^1(E_{2n},\mathbb{R})\big\}$. Next, we work with precycles. Just like (f_1,D), (f_2,D), $D = D_1 \times \cdots \times D_n$, we can also consider shuffling the variables $\{x_1,...,x_{2n}\}$. Thus for example, $\{1,...,2n\} = \{i_1,...,i_{2n}\}$, $f_1 = x_{i_1}^2 x_{i_2}\cdots x_{i_{2n-1}}^2 x_{i_{2n}} - 1$, $f_2 = x_{i_1}^2 x_{i_2}\cdots x_{i_{2n-1}}^2 x_{i_{2n}} - \mathrm{i}$, $D = D_{i_1}\times\cdots\times D_{i_n}$, $D_{i_j} \subset E_{i_{2j-1}} \times E_{i_{2j}}$, suitably interpreted so that $D \subset X$. We can construct a matrix Ξ, with $\binom{2n}{n}$ rows, defined by integration,

$$\int_D \log|f_j|\eta_\pm, \quad j = 1,2.$$

As we learned, each of these precycles can be naturally extended to K_1 classes on X. Now as E_j degenerates to $\{y_j^2 = x_j^3\}$, Ξ limits to Ξ° say. We deduce:

Theorem 6.3. *If rank $\Xi^\circ = \binom{2n}{n}$, then the primitive Hodge-$\mathcal{D}$-conjecture holds for general X. Namely,*

$$r_{n+1,1} : CH^{n+1}(X,1))\otimes\mathbb{R} \to \big\{H^1(E_1,\mathbb{R})\otimes\cdots\otimes H^1(E_{2n},\mathbb{R})(n)\big\}\bigcap H^{n,n}(X,\mathbb{R}(n)),$$

is surjective.

6.2. Applications to indecomposable K_1.

Definition 6.4. *A class $\{\xi\} \in CH^k(X,1)$ is said to be decomposable if one can choose $\xi = \sum_{j=1}^{N}(c_j, Z_j)$, where $c_j \in \mathbb{C}^\times \subset \mathbb{C}(Z_j)^\times$. The subgroup of decomposables is denoted by $CH^k_{dec}(X,1)$. The group of indecomposables is given by $CH^k_{ind}(X,1) := CH^k(X,1)/CH^k_{dec}(X,1)$.*

Corollary 6.5. *Let us assume that the regulator in (10) is nonzero, for very general $X := E_1 \times \cdots \times E_{2n}$, $n \geq 2$. Further, let $Y = E_1 \times \cdots \times E_{2n-1}$. Then $CH^{n+1}_{ind}(Y,1)$ is uncountable. This is the case when $n = 2, 3$.*

Proof. The proof actually is given in [GL] for $n = 2$. The same story if $n \geq 3$, using the same ideas, and which relies heavily on [Lew]. □

7. Appendix: Computer verification of the Hodge-$\mathcal{D}$-conjecture

In this appendix we provide a computer verification of the Hodge-$\mathcal{D}$ conjecture for a product of $2n$ elliptic curves, as described in the previous sections of this paper. We use MATLAB for the cases $n = 1, 2$ and SAGE for the general case.

7.1. The case $n = 1$.

In this case the matrix that describes the regulator is 2×2. From section 4, we already know that two of the entries are 0. We compute the other two.

```
fun11 = @(x,r) log(r.^2+2.*r.*sin(x)+1).*sin(x)./r
fun22 = @(x,r) log(r.^2+2.*r.*cos(x)+1).*cos(x)./r
a11 = -4.*(integral2(fun11,0,2*pi,0,1)+integral2(fun11,0,2*pi,1,10000000))
a22 = 4.*(integral2(fun22,0,2*pi,0,1)+integral2(fun22,0,2*pi,1,10000000))
```

The resulting matrix is

$$\begin{pmatrix} -16\pi & 0 \\ 0 & 16\pi \end{pmatrix} \tag{17}$$

Since this matrix has maximal rank, it follows that the Hodge-$\mathcal{D}$-conjecture is true for the product of two elliptic curves, as discussed in section 4.

7.2. The case $n = 2$.

The matrix of the regulator is now 6×6 (Notice that $\binom{2n}{n} = 6$ in this case). Henceforward, we'll use the $\eta_{i,\pm}$-base, as defined in section 5.

Let's start with D'', where $x_1x_2 = 1 = -y_1y_2$ and $x_3x_4 = 1 = -y_3y_4$. Since in this case $\mathrm{d}x_2 = -\frac{\mathrm{d}x_1}{x_1^2}$, we have $\eta^\circ_{1,+} = \eta^\circ_{1,-} = 0$, and the remaining η's are given by:

$$\begin{aligned}
\eta^\circ_{2,+} &= \omega_1 \wedge \omega_3 \wedge \bar\omega_2 \wedge \bar\omega_4 + \bar\omega_1 \wedge \bar\omega_3 \wedge \omega_2 \wedge \omega_4 \\
&= \frac{\mathrm{d}x_1}{y_1} \wedge \frac{\mathrm{d}x_3}{y_3} \wedge \frac{\mathrm{d}\bar x_2}{\bar y_2} \wedge \frac{\mathrm{d}\bar x_4}{\bar y_4} + \dots \\
&= \frac{\mathrm{d}x_1}{y_1} \wedge \frac{\mathrm{d}x_3}{y_3} \wedge \frac{\bar y_1 \mathrm{d}\bar x_1}{\bar x_1^2} \wedge \frac{\bar y_3 \mathrm{d}\bar x_3}{\bar x_3^2} + \dots \\
&= -\frac{\overline{y_1 y_3}}{y_1 y_3 \bar x_1^2 \bar x_3^2} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \\
&= -\frac{|y_1 y_3|^2}{x_1^3 x_3^3 \bar x_1^2 \bar x_3^2} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \qquad (18)\\
&= -\frac{|x_1|^3 |x_3|^3}{|x_1|^4 x_1 |x_3|^4 x_3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \\
&= -\frac{\overline{x_1 x_3}}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 - \frac{x_1 x_3}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 \\
&= \frac{-2\mathrm{Re}(x_1 x_3)}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\eta^\circ_{2,-} &= \frac{-2\mathrm{Im}(x_1 x_3)}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 \\
\eta^\circ_{3,+} &= \omega_1 \wedge \omega_4 \wedge \bar\omega_2 \wedge \bar\omega_3 + \bar\omega_1 \wedge \bar\omega_4 \wedge \omega_2 \wedge \omega_3 \\
&= \frac{\mathrm{d}x_1}{y_1} \wedge \frac{\mathrm{d}x_4}{y_4} \wedge \frac{\mathrm{d}\bar x_2}{\bar y_2} \wedge \frac{\mathrm{d}\bar x_3}{\bar y_3} + \dots \\
&= \frac{\mathrm{d}x_1}{y_1} \wedge \frac{y_3 \mathrm{d}x_3}{x_3^2} \wedge \frac{\bar y_1 \mathrm{d}\bar x_1}{\bar x_1^2} \wedge \frac{\mathrm{d}\bar x_3}{\bar y_3} + \dots \\
&= -\frac{\bar y_1 y_3}{y_1 \bar y_3 \bar x_1^2 x_3^2} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \\
&= -\frac{|\bar y_1 y_3|^2}{x_1^3 \bar x_3^3 \bar x_1^2 x_3^2} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \qquad (19)\\
&= -\frac{|\bar x_1|^3 |x_3|^3}{|x_1|^4 x_1 |x_3|^4 \bar x_3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 + \dots \\
&= -\frac{\bar x_1 x_3}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 - \frac{x_1 \bar x_3}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 \\
&= -\frac{2\mathrm{Re}(\bar x_1 x_3)}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3 \\
\eta^\circ_{3,-} &= -\frac{2\mathrm{Im}(x_1 \bar x_3)}{|x_1 x_3|^3} \mathrm{d}x_1 \wedge \mathrm{d}\bar x_1 \wedge \mathrm{d}x_3 \wedge \mathrm{d}\bar x_3
\end{aligned}$$

The following code gives the first two rows of the regulator matrix:

```
fun13 = @(x,y,r,s) log((r.*s).^2+1-2.*r.*s.*cos(x+y)).*cos(x+y)./(r.*s);
A=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun13(x,y,r,s),0,2*pi,0,2*pi),r,s),1.0001,6000,1.0001,6000);
B=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun13(x,y,r,s),0,2*pi,0,2*pi),r,s),0.0001,.999,0.0001,.999);
a13 = -2.*(B+A)
fun14 = @(x,y,r,s) log((r.*s).^2+1-2.*r.*s.*cos(x+y)).*sin(x+y)./(r.*s);
A=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun14(x,y,r,s),0,2*pi,0,2*pi),r,s),1.0001,6000,1.0001,6000);
B=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun14(x,y,r,s),0,2*pi,0,2*pi),r,s),0.0001,.999,0.0001,.999);
a14 = -2.*(B+A)
fun23 = @(x,y,r,s) log((r.*s).^2+1-2.*r.*s.*sin(x+y)).*cos(x+y)./(r.*s);
A=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun23(x,y,r,s),0,2*pi,0,2*pi),r,s),1.0001,6000,1.0001,6000);
B=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun23(x,y,r,s),0,2*pi,0,2*pi),r,s),0.0001,.999,0.0001,.999);
a23 = -2.*(B+A)
fun24 = @(x,y,r,s) log((r.*s).^2+1-2.*r.*s.*sin(x+y)).*sin(x+y)./(r.*s);
A=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun24(x,y,r,s),0,2*pi,0,2*pi),r,s),1.0001,6000,1.0001,6000);
B=integral2(@(r,s)arrayfun(@(r,s)integral2(@(x,y)fun24(x,y,r,s),0,2*pi,0,2*pi),r,s),0.0001,.999,0.0001,.999);
a24 = -2.*(B+A)
```

which translates to

$$\begin{pmatrix} 0 & 0 & 16\pi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16\pi^2 & 0 & 0 \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{pmatrix} \tag{20}$$

Notice that the degeneration obtained in the cases f_3, f_4, f_5, f_6 are given by permuting the indices $\{1, \dots, 6\}$. Therefore, the other rows of the matrix above can be deduced from the first two, giving the following matrix:

$$\begin{pmatrix} 0 & 0 & 16\pi^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16\pi^2 & 0 & 0 \\ 16\pi^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16\pi^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16\pi^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16\pi^2 \end{pmatrix} \tag{21}$$

Since this matrix has maximal rank, it follows that the Hodge-$\mathcal{D}$-conjecture is true for the product of four elliptic curves, as discussed in section 5.

7.3. **The general case.** The non-triviality of the regulator is a consequence of statement 6.1, which depends on the integrals (13) and (16). The following SAGE code computes the integral (13) and (16) given n and m:

```
def int_int(n,m):
    sins=1
    sums=0
    allvar=[]
    for i in range(n):
        allvar.append(var('x_%d' % i))
        sins = sins*sin(allvar[i])
        sums = sums + allvar[i]
    #print(sins)
    #print((cos(sums))^m)
    f=sins*(cos(sums))^m
    #g=sins*(sin(sums))^m
    i0=integral(f, allvar[0], 0,2*pi)
    if n>1:
        for i in range(n-1):
            i0 = integral(i0, allvar[i+1],0,2*pi)
            print(i,i0)
    #print(i0)
    return i0
```

We've checked both (i) and (ii) from Statement 6.1, when $n \leq 100$ and $m = 1$. In fact, we have:

Proposition 7.1. *If $n \leq 100$ and $m = 1$, then:*

a) *If $n = 2k$, the integral 6.1 (i) is $(-1)^k\pi^n$ and 0 otherwise.*
b) *If $n = 2k+1$, the integral 6.1 (ii) is $(-1)^k\pi^n$ and 0 otherwise.*

Using the $\eta_{i,\pm}$ base and the previous discussion, we know it suffices to compute the first two rows of the regulator matrix in order to determine its rank.

Moreover, each term of the the first row is one of the following type

$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \log((r_1 r_2 \ldots r_n)^2 + 1 - 2\cos(x_1 + \ldots + x_n)) \cos(\pm x_1 \pm x_2 \ldots \pm x_n) \mathrm{d}A$$
$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \log((r_1 r_2 \ldots r_n)^2 + 1 - 2\cos(x_1 + \ldots + x_n)) \sin(\pm x_1 \pm x_2 \ldots \pm x_n) \mathrm{d}A \tag{22}$$

and each term of the second row is one of the following

$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \log((r_1 r_2 \ldots r_n)^2 + 1 - 2\sin(x_1 + \ldots + x_n)) \cos(\pm x_1 \pm x_2 \ldots \pm x_n) \mathrm{d}A$$
$$\int_0^\infty \cdots \int_0^\infty \int_0^{2\pi} \cdots \int_0^{2\pi} \log((r_1 r_2 \ldots r_n)^2 + 1 - 2\sin(x_1 + \ldots + x_n)) \sin(\pm x_1 \pm x_2 \ldots \pm x_n) \mathrm{d}A \tag{23}$$

The above integrals can be computed using the following code:

```
n=3
sums=0
prods=1
allvar=[]
for i in range(n):
    allvar.append(var('x_%d' % i))
    sums = sums + allvar[i]
    #sums = sums + (-1)**(i+1) * allvar[i]
for i in range(n,2*n):
    allvar.append(var('x_%d' % i))
    prods = prods*allvar[i]
#print(prods)
#print(cos(sums))
f=log((prods)**2-2*prods*cos(sums)+1)*(cos(sums)/(prods))
#g=log((prods)**2-2*prods*cos(sums)+1)*(sin(sums)/(prods))
i0=integral(f, allvar[0], 0,2*pi, algorithm="giac")
if n>1:
    for i in range(n-1):
        i0 = integral(i0, allvar[i+1],0,2*pi)
print(i0)
```

Remark 7.2. *For large n, the code from this appendix should take some time to compile, and needs to be adjusted depending on which term of the regulator matrix is to be found.*

Remark 7.3. *MATLAB performs poorly if $n >> 0$ when compared to SAGE. On the other hand, for $n < 5$ MATLAB is more accurate for numerical integration.*

Remark 7.4. *The algorithm used with the integration package of SAGE was GIAC. MAXIMA and SCIPY tend to crash even for small n.*

References

[B] S. Bloch, *Algebraic cycles and higher K-theory,* Adv. Math. **61**, (1986), 267-304.

[C-L1] X. Chen, J. D. Lewis, *The Hodge-$\mathcal{D}$-conjecture for $K3$ and Abelian surfaces,* J. Algebraic Geom. **14** (2005), 213-240.

[C-L2] ______, *Noether-Lefschetz for K_1 of a certain class of surfaces,* Bol. Soc. Mat. Mexicana **10(3)** (2004), 29-41.

[C-L3] ______, *The real regulator for a product of $K3$ surfaces,* AMS/IP Studies in Advanced Mathematics, Volume **38** (2006), 271-283.

[GL] B. B. Gordon, J. D. Lewis, *Indecomposable higher Chow cycles on products of elliptic curves,* J. Algebraic Geom. **8** (1999), 543-567.

[Lew] J. D. Lewis, *A note on indecomposable motivic cohomology classes,* J. Reine Angew. Math. **485** (1997), 161-172.

[Ma] K. Mansour, Thesis (in progress).

[MS] S. Müller-Stach, *Constructing indecomposable motivic cohomology classes on surfaces,* J . Algebraic Geom. **6** (1997), 513-543.

[T] I. Turkmen, *Regulator indecomposable cycles on a product of elliptic curves,* Canadian Mathematical Bulletin **56**(3) (2013), 640-646.

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