Exercises

- Given two natural numbers $a, b \in \mathbb{N}$, prove that there is a natural number $m \in \mathbb{N}$ such that $m \cdot a > b$.

  Solution. Suppose not, then the set $X = \{m \cdot a\,;\, m \in \mathbb{N}\}$ would be bounded, hence finite; a contradiction. □

- Let $a \in \mathbb{N}$. If the set X has the following property: $a \in X$ and $n \in X \Rightarrow n + 1 \in X$. Then X contains all natural numbers greater than or equal to a.

  Solution. The claim is that $a + n \in X$ for every $n \in \mathbb{N}$. If $n = 1$, $a \in X$ by hypothesis. Suppose $a + n \in X$, then $(a + n) + 1 \in X$ and the result follows by induction. □

- A number $a \in \mathbb{N}$ is called **predecessor** of $b \in \mathbb{N}$ if $a < b$ and there is no $c \in \mathbb{N}$ such that $a < c < b$. Prove that every number, except 1, has a predecessor.

  Solution. Notice that if a is the predecessor of b then $b = a + 1$. The result follows then follows from the fact that the sucessor function is surjective. □

- Give an example of a surjective function $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$, the set $f^{-1}(n)$ is infinite.

  Solution. We proved in class that $\mathbb{N} \times \mathbb{N}$ is countable, so there is a bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Consider the projection $\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ given by $\pi(x, y) = x$. Then $\pi \circ f$ is an example. □

- Show that if A is countably infinite then $\mathcal{P}(A)$ is uncountable. **(This question was typed wrong! This is the version that makes sense.)**

  Solution. Notice that $\mathcal{P}(A) = \mathcal{F}(A; \{0, 1\})$. Cantor's theorem gives that $\mathcal{P}(A)$ is uncountable. □

- (Cantor-Bernstein-Schroder theorem) Given sets A and B, let $f : A \to B$ and $g : B \to A$ be injective functions. Show that there is a bijection $h : A \to B$.

  Solution. Notice that given $x \in A$, after successive applications of f and g we produce a path that either lands back at x, or doesn't. In the former case, set $h(x) = f(x)$. If we don't land back at x, we have an infinite path starting at x or containing x. If the path starts in A, set $h(x) = f(x)$, whereas if the path starts in B, set $h(x) = g^{-1}(x)$. If the path if infinite, contains x but is not cyclic, set $h(x) = f(x)$. The functions h is a bijection by construction. □