Exercises

1. If $\lim x_n = a$, show that $\lim |x_n| = |a|$. Show that the converse can be false by giving a counter example.

   Solution. We can find n_0 such that $n > n_0 \Rightarrow |x_n - a| < \epsilon$. Hence,

   $$||x_n| - |a|| \leq |x_n - a| < \epsilon,$$

   which proves $\lim |x_n| = |a|$. For the counterexample take, say $x_n = (-1)^n$. $\square$

4. Given an example of a sequence x_n and a infinite decomposition of $\mathbb{N} = \mathbb{N}_1 \cup \ldots \cup \mathbb{N}_k \cup \ldots$, such that for every $k \in \mathbb{N}$, the subsequence $(x_n)_{n \in \mathbb{N}_k}$ has limit $a \in \mathbb{R}$ but $\lim x_n \neq a$.

   Solution. Let $\mathbb{N}_k = \{n; n = 2^{k-1} m, \text{ for odd } m\}$ and set $x_{n_k} = 1$ if $n_k = \min\{\mathbb{N}_k\}$, and $x_{n_k} = \frac{1}{n_k}$ otherwise. Then $\lim x_{n_k} = 0$, regardless of the k. But $\lim x_n$ doesn't exits since x_n has a constant subsequence equal to 1. $\square$

6. Show that $(1 - \frac{1}{n})^n$ is increasing. *Hint: Use the inequality of arithmetic and geometric means involving the* $n+1$ *numbers* $1 - \frac{1}{n}, \ldots, 1 - \frac{1}{n}, 1$.

   Solution. The hint literally is the answer. $\square$

12. Let $x_1 = 1$ and $x_{n+1} = 1 + \sqrt{x_n}$. Show that x_n is bounded and find $\lim x_n$.

    Solution. Notice that $x_{n+1} - x_n = \sqrt{x_n} - \sqrt{x_{n-1}}$, and that $x_2 > x_1 = 1$. By induction, $x_{n+1} > x_n$ and the sequence x_n is increasing. We have $x_n < x_{n+1} = 1 + \sqrt{x_n}$, hence $(x_n - 1)^2 < x_n$ or $x_n^2 - 3x_n + 1 < 0$. This can only happen if $x_n < \frac{3+\sqrt{5}}{2}$. Hence x_n is bounded, so it converges because it's monotone, say $x_n \to a$. Taking the limit in the recursion we have that a satisfies $a^2 - 3a + 1$, since $a > 1$, the only possibility is $a = \frac{3+\sqrt{5}}{2}$. $\square$

13. Show that x_n doesn't have a convergent subsequence if and only if $\lim |x_n| = +\infty$.

    Solution. Suppose x_n doesn't have a convergent subsequence, if $|x_n|$ was convergent then it would be bounded and hence by Bolzano-Weierstrass, it would have a converging subsequence. Conversely, suppose $|x_n| \to +\infty$, then any subsequence would be unbounded, hence divergent. $\square$