Exercises

1. Let $f, g, h : X \to \mathbb{R}$ be functions such that, for every $x \in X$ we have $f(x) \leq g(x) \leq h(x)$. Show that if there is a point $a \in X \cap X'$ such that $f(a) = h(a)$ and $f'(a) = h'(a)$ then $g'(a)$ exists and $g'(a) = f'(a) = h'(a)$.

   Solution. Squeeze theorem since f(a)=g(a)=h(a). □

2. Let $p : \mathbb{R} \to \mathbb{R}$ be an odd degree polynomial. Then there exists $c \in \mathbb{R}$ such that $p''(c) = 0$.

   Solution. IVT and the fact that derivative is still polynomial. □

3. Let $f : X \to \mathbb{R}$ be differentiable at $a \in X \cap X'$. If x_n and y_n are sequences in X such that $\lim x_n = \lim y_n = a$ and $x_n < a < y_n$ for every $n \in \mathbb{N}$, show that

$$\lim \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

   Solution. Write $\frac{f(y_n)-f(x_n)}{y_n-x_n} = t_n \frac{f(y_n)-f(a)}{y_n-a} + (1-t_n)\frac{f(x_n)-f(a)}{x_n-a}$ where $t_n = \frac{y_n-a}{y_n-x_n} \leq 1$ and use Squeeze. □

4. Show that the function given by $f(0) = 0$, $f(x) = x^2 \sin \frac{1}{x}$ if $x \neq 0$, is differentiable. Find sequences x_n and y_n such that $x_n \neq y_n$, $\lim x_n = \lim y_n = 0$ but $\lim \frac{f(y_n)-f(x_n)}{y_n-x_n}$ doesn't exist.

   Solution. f is clearly diff. at $x \neq 0$. At 0 observe that $f'(0) = \lim \frac{f(x)}{x}$ and use Squeeze. Take $y_n := \frac{1}{\frac{\pi}{2}n}$ and $x_n := \frac{1}{\frac{\pi}{2}(n+1)}$. □

5. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. We call $a \in I$ a *critical point* if $f'(a) = 0$. We say a critical point a is *non-degenerate* if $f''(a) \neq 0$.

   5.1 If $f \in C^1$, show that the set of all critical points contained in a closed interval $[c, d] \subseteq I$ is closed.

   Solution. Let $X = \{a \in [c, d]; f'(a) = 0\}$ and $a_n \in X$. Suppose $a_n \to a$ ($a \in [c, d]$ since the former is closed), since f' is continuous, $f'(a_n) \to f'(a)$. Since $f'(a_n) = 0$, this implies $f'(a) = 0$, hence $a \in X$. □

   5.2 Show that local maximum and minimum points of f are critical points. Moreover, any critical non-degenerate point is a maximum or minimum.

   Solution. Only the second claim is not clear. Suppose $f'(c) = 0$ and $f''(c) > 0$, then $\exists \delta > 0$ such that $c - \delta < x < c < y < c + \delta \Rightarrow f'(x) < 0 < f'(y)$ □

5.3 Show that there are C^∞ functions with isolated degenerate local maximum/minimums. Moreover, there are critical points of C^∞ functions that are not local maximum/minimum points.

Solution. $f(x) = x^4$;$g(x) = x^3$ at 0 □

5.4 Show that every non-degenerate critical point of f is isolated.

Solution. $f''(c) \neq 0 \Rightarrow \lim_{x\to c} \frac{f'(x)-f'(c)}{x-c} \neq 0 \Rightarrow f'(x) \neq 0$ □

5.5 Let $f \in C^1$, suppose that the critical points of f contained in a closed interval $[c, d] \subseteq I$ are non-degenerate. Show that there are finitely many of them. Conclude that f has at most a countable number of non-degenerate critical points in I.

Solution. use 5.4; $\mathbb{R} = \cup_n [n, n+1]$ □

5.6 The function $f(0) = 0$, $f(x) = x^4 \sin \frac{1}{x}$ if $x \neq 0$ has infinitely many non-degenerate critical points in $[0, 1]$. Wouldn't this be a contradiction to 5.4? Why/why not?

Solution. No, f isn't C^1 □

6. Let $f : I \to \mathbb{R}$ be a function defined on interval $I \subseteq \mathbb{R}$. If there is $C, \alpha > 0$ such that $\forall x, y \in I \Rightarrow |f(x) - f(y)| \leq C|x - y|^\alpha$, we say f is *Holder continuous*. Show that if $\alpha > 1$ then f is constant.

   Solution. Take $\alpha = 1 + \beta$. Fix $a \in I$ Then $\frac{|f(x)-f(a)|}{|x-a|} \leq C|x - a|^\beta$ and take the limit $x \to a$. □

7. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. Show that if $f'(x) = 0$ for every $x \in I$ then f is constant.

   Solution. MVT applied to $[a, x]$ □

8. Show that a differentiable function $f : I \to \mathbb{R}$ is Lipschitz, i.e. $|f(x)-f(y)| \leq C|x-y|$, if and only if $|f'(x)| \leq C$.

   Solution. implication trivial; converse is by proof by contradiction □

9. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $f \in C^\infty$, $f(x) \neq x$, $\forall x \in \mathbb{R}$ and $|f'(x)| < 1$.

   Solution. $\ln(2 + x^2)$ □

10. Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \cos(\cos(x))$. Show that $|f'(x)| \leq c < 1$ for some $c \in \mathbb{R}$.

Solution. $f'(x) = \sin(x)\sin(\cos x) \Rightarrow |f'(x)| \le |\sin(\cos x)|$ and $\sin(\cos x)$ is decreasing on $[0, \pi]$ hence $|f'(x)| \le \sin 1 < 1$ □

11. Let $f : (a, +\infty) \to \mathbb{R}$ be differentiable. Show that if $\lim_{x\to+\infty} f(x) = b$ and $\lim_{x\to+\infty} f'(x) = c$, then $c = 0$. *[Hint: Apply the Mean Value theorem on $[n, n+1]$ and let $n \to +\infty$.]*

12. Let $f : [a, b] \to \mathbb{R}$ be continuous, differentiable on (a, b), satisfying $f(a) = f(b)$. Given $k \in \mathbb{R}$, show that $\exists c \in (a, b)$ such that $f'(c) = kf(c)$. *[Hint: Apply Rolle's theorem to $g(x) = f(x)e^{-kx}$.]*

13. Let $f : I \to \mathbb{R}$ be differentiable on an interval $I \subseteq \mathbb{R}$. A *root* of f is a number $c \in I$ such that $f(c) = 0$. Show that between two consecutives roots of f', there is at most one root of f.

    Solution. By contradiction and Rolle's theorem. □

14. Let $f : [0, +\infty) \to \mathbb{R}$ be twice differentiable. Show that if f'' is bounded and $\lim_{x\to+\infty} f(x)$ exists, then $\lim_{x\to+\infty} f'(x) = 0$.

    Solution. By MVT: $f(n+1) = f(n) + f'(x_n)$ □

15. Show that the composition of C^k functions is still a C^k function.

    Solution. trivial □

16. Given $a, b \in \mathbb{R}$ with $a < b$, consider $\varphi : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi(x) = \begin{cases} e^{\frac{1}{(x-a)(x-b)}}, & \text{if } x \in (a, b), \\ 0, & \text{if } x \notin (a, b). \end{cases}$$

    Show that $\varphi \in C^\infty$ and φ has exactly one maximum point.

    Solution. $\varphi'(x) = e^{\frac{1}{(x-a)(x-b)}} \frac{-1}{[(x-a)(x-b)]^2}[2x - a - b]$, hence $\varphi'(x) = 0 \Rightarrow x = \frac{a+b}{2}$ □

17. Let $f : I \to \mathbb{R}$ be twice differentiable at $a \in I^\circ$. Show that

$$f''(a) = \lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2}$$

    Given a example where the limit above exists but $f'(a)$ doesn't.

    Solution. By Taylor:

$$\begin{aligned} f(a+h) &= f(a) + f'(a)h + \frac{f''(a)}{2}h^2 + r(h) \\ f(a-h) &= f(a) - f'(a)h + \frac{f''(a)}{2}h^2 + s(h) \end{aligned} \tag{1}$$

    ; example: $\sin\frac{1}{x}$ □

18. Show that the function $f(x) = |x|^{2n+1}$ is of class C^{2n} but $f^{(2n+1)}(x)$ doesn't exist.