On the arithmetic of Landau-Ginzburg model of a certain class of 3-folds[1]

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[1]Working in progress with Prof. Charles Doran and Andrew Harder from University of Alberta

1.Introduction

Notation: We denote Laurent Polynomials by $f(x_1, x_2, x_3) \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ and write $\mathbf{x}^{\mathbf{v}} = x_1^a x_2^b x_3^c$ for $\mathbf{v} = (a, b, c)$

Definition

Let Δ be a three-dimensional reflexive polytope in $\mathbb{R}^3$. Let f be a Laurent polynomial in three variables. We say that f is a three-dimensional **Minkowski polynomial** with Newton polytope Δ if:

- Newton(f)= Δ
- For each facet Q of Δ we have:

$$f_{Q:Q_1,Q_2,\ldots,Q_n} = \sum_{v \in Q \cap \mathbb{Z}^3} a_v \mathbf{x}^{\mathbf{v}}$$

  for some admissible lattice Minkowski Decomposition $Q = Q_1 + Q_2 + \cdots + Q_n$ and f_Q is the facet polynomial to be defined below.

We can construct Minkowski polynomials by specifying the "face" polynomials of Δ in the following way: To each line segment L from p to q, we set $f_L = \mathbf{x^p} + \mathbf{x^q}$ and for each triangle T_n with vertices u, v, w such that $T_n \cap \mathbb{Z}^3 = \{u, v = v_0, \ldots, v_n = w\}$, we set $f_{T_n} = \mathbf{x^u} + \sum_{k=0}^{n} \binom{n}{k} \mathbf{x^{v_k}}$. If a face has a lattice Minkowski decomposition $F = F_1 + F_2 + \cdots + F_n$, then we set $f_{F:F_1,F_2,\ldots,F_n} = \Pi f_{F_i}$. So a lattice Minkowski decomposition completely determines the Minkowski polynomial.

Example

Let $\Delta = \mathbf{conv}(e_1, e_2, e_3, -e_1 - e_2 - e_3)$, the face polynomials are $x + y + z$, $x + y + (xyz)^{-1}$, $x + z + (xyz)^{-1}$,$z + y + (xyz)^{-1}$. It follows from the definition that the Minkowski Polynomial is $f = x + y + z + (xyz)^{-1}$.

Definition

We shall say that a linear homogeneous recurrence R with polynomial coefficients is a recurrence of the Apery type, if there is a Dirichlet character with L-function $L(s)$, an argument $s_0 \in \mathbb{Z}$, $s_0 > 1$ and two solutions of R, a_n, b_n, such that:

$$\lim \frac{a_n}{b_n} = cL(s_0), c \in \mathbb{Q}^*$$

The limit above is called **Apery limit** of R. For our purposes, the solutions a_n, b_n will satisfy $a_0 = 1$ and b_n is the unique solution with $b_0 = 0, b_1 = 1$.

Example

This example is due Apery. He proved the irrationality of $\zeta(3)$ by considering the recurrence:

$$n^3 u_n - (34n^3 - 51n^2 + 27n - 5)u_{n-1} + (n-1)3u_{n-2} = 0$$

If a_n is the solution with $a_0 = 1, a_1 = 5$,and b_n the solution with $b_0 = 0, b_1 = 1$, then he proved that:

$$|\zeta(3) - \frac{6b_n}{a_n}| = o(a_n^{-2})$$

Therefore, $\frac{\zeta(3)}{6}$ is the Apery limit of the recurrence above.

We now consider the following Fano Picard rank 1 threefolds:

V_{12} := A section of the orthogonal Grassmannian $O(5,10)$ by a codimension 7 plane
V_{16} := A section of the Lagrangian Grassmannian $L(3,6)$ by a codimension 3 plane
V_{18} := A section of G_2/P by a codimension 2 plane

Using a method[2] by Golyshev, we can produce **Picard-Fuchs operators**(and therefore recurrences from $D_{PF} \cdot X = 0$) for each one of the above Fanos. The Apery limit[3] for each one of these is:
$\frac{1}{6}\zeta(3)$ for V_{12}
$\frac{7}{6}\zeta(3)$ for V_{16}
$\frac{1}{3}L(\chi_3, 3)$ for V_{18}

[2] Vasily V. Golyshev, Classification problems and mirror duality.,Surveys in geometry and number theory.London Mathematical Society Lecture Note Series 338, 88-121 (2007)., 2007.

[3] Golyshev,V., Deresonating a Tate period. arXiv:0908.1458

2.Landau-Ginzburg models

Mirror symmetry relates a Fano variety with a dual object called Landau-Ginzburg model, which is a variety equipped with a non-constant complex valued function.For example, a LG model for $\mathbb{P}^2$ is a family of elliptic curves and more generaly, the LG model of a Fano n-fold is a family of Calabi-Yau $(n-1)$-folds. In general, mirror symmetry relates sympletic properties of a Fano variety with algebraic ones of the mirror and virce versa.

Using the notation above, let $\mathbb{P}_{\Delta^\circ}$ be a Toric degeneration of V_k, then each one of these will have a mirror Landau-Ginzburg model, which is a family of $K3$ surfaces in $\mathbb{P}_\Delta$, that can be constructed as follows: Let f be a Minkowski polynomial for Δ, then the family of $K3$s is:

$$X_t := \overline{\{1 - tf(\mathbf{x}) = 0\}} \subset \mathbb{P}_\Delta$$

Let $\omega_t = Res_{X_t}\left(\frac{\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2}\wedge\frac{dx_3}{x_3}}{1-tf}\right)$ and γ_t a vanishing cycle. We define the period of f by:

$$\Pi_f(t) = \int_{\gamma_t}\omega_t = \sum a_n t^n$$

Where a_n is the constant term of f^n. We say that a_n is the period sequence of f.

Consider a polynomial differential operator $L = \sum t_k P_k(D)$ where $P_k(D)$ is a polynomial in $D = t\frac{d}{dt}$, then $L \cdot \Pi_f(t) = 0$ is equivalent to a linear recursion relation. In practice, to compute L one uses knowledge of the first few terms of the period sequence and linear algebra to guess the recursion relation. The operator L is called **Picard Fuchs operator**.

Example

Periods of the Mirror Quintic. The period sequence for the Mirror quintic is $a_n = \frac{(5n)!}{n!^5}$; it satisfies the following recurrence:

$$(n+5)^5 a_{n+1} = (5n+5)(5n+1)(5n+2)(5n+3)(5n+4)a_n$$

The Picard-Fuchs operator is easily seen to be:

$$D^4 - 5t(5D+5)(5D+1)(5D+2)(5D+3)(5D+4)a_n$$

In this talk we will prove that the Apery limits described above for each one of the V_k, have a motivic meaning, more precisely we prove that these Apery constants are special values of higher normal functions arising from higher regulators:

Theorem

For each Fano V_{12}, V_{16} and V_{18} described above, there is a higher normal function V such that the constants above are equal to $V(0)$

The rest of the talk will be dedicated to prove this theorem. The first thing to do is to prove the following result:

Proposition

Every 3 dimensional Minkowski polynomial f is tempered,i.e the higher Chow cycle $\{x_1, x_2, x_3\}_{X_t^*}$ vanish under the Tame symbol map.(X_t^* denotes the intersection of X_t with $\mathbb{P}_\Delta \setminus (\mathbb{C}^*)^3$)

Proof

For each X_t, we set $D = \bigcup D_\sigma$ to be the the intersection of the $K3$ with the toric boundary. Then the we have the localization exact sequence:

$$\cdots \to CH^2(D,3) \to CH^3(X_t,3) \to CH^3(X_t \backslash D, 3) \xrightarrow{Res} CH^2(D,2) \to \dots$$

where $Res = \bigoplus_i Tame_i$.
Also, the following sequence is exact:

$$0 \to CH^2(D,2) \to \bigoplus_i CH^2(D_i \backslash \cup D_i \cap D_j, 2) \to \bigoplus_{i,j} CH^1(D_i \cap D_j, 1)$$

Here the D_i are the irreducible components of D that we get

Proof(cont.)

when setting the facet polynomial equals 0.
By the above sequence we have:

$$CH^2(D_i,2)=Ker\{CH^2(D_i\setminus\cup D_i\cap D_j,2)\xrightarrow{Res}\bigoplus_j CH^1(D_i\cap D_j,1)\}$$

Now if for every i,j, the composition:

$$CH^3(X_t\setminus D,3)\xrightarrow{Res_i}CH^2(D_i\setminus\cup D_i\cap D_j,2)\xrightarrow{Res_j}CH^1(D_i\cap D_j,1)$$

sends ξ to 0, so $Res_i\xi\in CH^2(D_i,2)$. Now in dimension 3, the irreducible pieces of a lattice Minkowski decomposition are either segments or triangles as was defined above, hence all the D_i are rational and therefore $CH^2(D_i,2)\cong K_2(\mathbb{C})$, and if working over $\overline{\mathbb{Q}}$ we have:

$$CH^2(D_i,2)\cong K_2(\overline{\mathbb{Q}})=0$$

Proof(cont.)

Therefore $Res_i\xi$ is torsion, in particular the class $Res_i\{x_1, x_2, x_3\}_{X_t^*}$ is torsion and it follows that the Minkowski polynomial is tempered. $\square$

Since f is tempered, the family of higher Chow cycles lifts to X_t, yielding[4] a family of motivic cohomology classes $[\Xi_t]$(in a smooth fiber this is just higher Chow cycles) on the Landau-Ginzburg model, i.e in $H^3_{\mathcal{M}}(X^\lambda, \mathbb{Q}(3))$,where we set $X^\lambda := X_t$ for $\lambda = \frac{1}{t}$. Therefore if AJ is the Abel-Jacobi map[5], then:

$$AJ([\Xi_\lambda]) \in H^2(X^\lambda, \mathbb{C}/\mathbb{Q}(3))$$

Let $\mathcal{R}_\lambda$ be any lift of this class to $H^2(X^\lambda, \mathbb{C})$, then if $\gamma_\lambda \in H_2(X^\lambda, \mathbb{C})$, it's clear that the pair $\langle \mathcal{R}_\lambda, \gamma_\lambda \rangle$ makes sense.

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[4] Kerr,M; Doran,C. Algebraic K-theory of toric hypersurfaces, CNTP 5 (2011), no. 2, 397-600, Theorem 3.8

[5] In smooth fibers, AJ takes a rather simple form in terms of currents, see M. Kerr, J. Lewis, and S. Mller-Stach, The Abel-Jacobi map for higher Chow groups,Compos. Math. 142 (2006), no. 2, 374-396

Set $\tilde{\omega_t} = t\omega_t$, for V_k considered above, the degeneration[6] of X^λ is of **type III**(Maximal unipotent monodromy, i.e Hodge Tate Limiting mixed Hodge structure), then we can define $\widetilde{\Pi_f(t)} = \int_{\beta_t} \tilde{\omega_t}$, where β_t is a vanish cycle around $\lambda = 0$. Using Golyshev's results[7] we have that:

$$\widetilde{\Pi_f}(\lambda) = \Pi_f(\lambda)$$

And the Picard-Fuchs operator $\widetilde{D_{PF}}$ is just D_{PF} with t replaced by λ.

[6] A.N. Parshin, I.R. Shafarevich (Eds.) Algebraic Geometry V. Fano Varieties. Series: Encyclopaedia of Mathematical Sciences, Vol. 47

[7] Golyshev, V.,Deresonating a Tate period. arXiv:0908.1458.

Under these assumptions and using the notation from above, we define:

$$V(\lambda) := \langle \mathcal{R}_\lambda, \tilde{\omega_\lambda} \rangle$$

It is known[8] that this defines a single valued function on a disc about $\lambda = 0$, also: $\widetilde{D_{PF}} V(\lambda) = \{\text{symbol of } D_{PF}\} \times \{\text{Yukawa coupling}\} = k\lambda$, $k \in \mathbb{Z}$.
Set $\widetilde{\Pi_f}(\lambda) = \sum_{n \geq 0} a_n \lambda^n$, as remarked above, $\widetilde{D_{PF}} \cdot \widetilde{\Pi_f}(\lambda) = 0$ is equivalent to a linear recurrence equation having a_n as a solution.
Let $P_f(\lambda) := -V(\lambda) + \widetilde{\Pi_f}(\lambda) V(0) = \sum_{n \geq 1} b_n \lambda^n$, note that b_n is a solution for the same recurrence. We have that:

$$V(\lambda) = \sum (a_n V(0) - b_n) \lambda^n$$

So $|a_n V(0) - b_n| \to 0$ and $|a_n|, |b_n| \to \infty$, hence $\frac{b_n}{a_n} \to V(0)$.

[8] Kerr,M; Doran,C. Algebraic K-theory of toric hypersurfaces, CNTP 5 (2011), no. 2, 397-600, §4.

Therefore, $V(0)$ is the Apery limit for the recurrence obtained by the period sequence of $\widetilde{D_{PF}}$. On the other hand, $H^3_{\mathcal{M}}(X^0, \mathbb{Q}(3))$ has a subgroup $H^1_{\mathcal{M}}(Spec(\mathbb{C}), \mathbb{Q}(3))$ which is indecomposable $K_5(\mathbb{Q})$ so the Abel-Jacobi map restricted to this subgroup is the Borel regulator. One can prove that $V(\lambda)$ limits to a value of the Borel regulator, by Borel's theorem it has to be(modulo $\mathbb{Q}(3)$) a multiple of $\zeta(3)$. Hence $V(0)$ modulo a rational number is indeed equals to the Apery constant for each V_k considered above.

What's next?

- The next obvious cases to check are of course the other Picard rank one Fano threefolds similar to these, namely, those that are complete intersection in Grassmanians of simple Lie groups other than projective spaces: V_{10} and V_{12}. The argument above is not valid in those cases since the Monodromy for those is not maximal unipotent[9], in fact is not even unipotent! Also, with the construction above, one ends up with a multiple of $\zeta(3)$ or a torsion value in $\mathbb{Q}(3)$, none of which can be a multiple of $\zeta(2)$, the Apery limit in those cases. We[10] are in the process of describing the Apery constants in those cases too.

[9] A.N. Parshin, I.R. Shafarevich (Eds.) Algebraic Geometry V. Fano Varieties. Series: Encyclopaedia of Mathematical Sciences, Vol. 47

[10] Joint work with Prof. Charles Doran and Andrew Harder from University of Alberta

What's next?

- Another interesting thing that we will describe is the analysis of the other 101 cases of Fano threefolds.
- The Higher dimension generalization is an interesting non trivial problem:
  Conjecture: For $n \geq 4$, the Apery limit of a Landau-Ginzburg model of a Fano n-fold can be seen as a special value of a Higher normal function arising from a higher regulator of the toric symbol $\{x_1, x_2, \ldots, x_n\}$
- In fact, in dimension greater than 4, we can talk about more than one "Apery constants", basically the question is to describe the relationship between primitive quantum cohomology classes in the higher dimensional Fano and the Landau-Ginzburg model of them.

Thanks!