HOMEWORK 2

(1) Show that the tangent bundle of a manifold is Hausdorff.

(2) Let $(U, \phi) = (U, x_1, \ldots, x_n)$ and $(V, \psi) = (V, y_1, \ldots, y_n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \bar{\phi})$ and $(TV, \bar{\psi})$ on the total space TM of the tangent bundle, with transition function $\bar{\psi} \circ \bar{\phi}^{-1}$:

$$(x_1, \ldots, x_n, a_1, \ldots, a_n) \to (y_1, \ldots, y_n, b_1, \ldots, b_n)$$

(a) Compute the Jacobian matrix of the transition function $\bar{\psi} \circ \bar{\phi}^{-1}$ at $\phi(p)$.
(b) Show that the Jacobian determinant of the transition function $\bar{\psi} \circ \bar{\phi}^{-1}$ at $\phi(p)$ is $(det[\frac{\partial y_i}{\partial x_j}])^2$.

(3) Suppose M and N are smooth manifolds with M connected, and $F : M \to N$ is a smooth map such that $F_* : T_pM \to T_{F(p)}N$ is the zero map for each $p \in M$. Show that F is a constant map.

(4) Show that there is a smooth vector field on $\mathbb{S}^2$ that vanishes at exactly one point. [Hint: Try using stereographic projection.]

(5) Show that $\mathbb{T}^n = \mathbb{S}^1 \times \ldots \mathbb{S}^1$ is parallelizable, i.e $T\mathbb{T}^n$ is the trivial bundle. [Hint: For example, on $\mathbb{T}^3$, consider the vector fields:

$$\begin{aligned} X_1 &= -x\frac{\partial}{\partial w} + w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} \\ X_2 &= -y\frac{\partial}{\partial w} + z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} - x\frac{\partial}{\partial z} \\ X_3 &= -z\frac{\partial}{\partial w} - y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \end{aligned} \tag{0.1}$$