HOMEWORK 3

(1) The length of a smooth curve $\gamma : [a, b] \to \mathbb{R}^n$ is defined to be the value of the (ordinary) integral

$$L(\gamma) = \int_a^b |\gamma'(t)| dt$$

Show that there's no smooth covector field ω on $\mathbb{R}^n$, with the property that $\int_\gamma \omega = L(\gamma)$ for every curve γ.

(2) Suppose X is a smooth vector field on an open set $U \in \mathbb{R}^n$, thought of as a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^n$. For any piecewise smooth curve segment $\gamma : [a, b] \to U$, define the line integral of X over γ by

$$\int_\gamma X \cdot ds = \int_a^b X(\gamma(t)) \cdot \gamma'(t) dt$$

and say X is conservative if its line integral around any closed curve is zero.

(a) Show that X is conservative if and only if there exists a smooth function $f \in C^\infty(U)$ such that $X = \nabla f$, the gradient of f. [Hint: Consider the covector field $\omega_x(Y) = X(x) \cdot Y$, where the dot denotes the Euclidean dot product.]

(b) If $n = 3$ and X is conservative, show $curl\, X = 0$, where

$$curl\, X = \left(\frac{\partial X_3}{\partial x_2} - \frac{\partial X_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \left(\frac{\partial X_1}{\partial x_3} - \frac{\partial X_3}{\partial x_1}\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial X_2}{\partial x_1} - \frac{\partial X_1}{\partial x_2}\right)\frac{\partial}{\partial x_3}$$

(c) If $U \in \mathbb{R}^3$ is star-shaped, show that X is conservative on U if and only if $curl\, X = 0$.

(3) If M is a compact manifold and $f \in C^\infty(M)$, show that df vanishes somewhere on M.

(4) If $F : M \to N$ is a smooth map, show that $F^* : T^*N \to T^*M$ is smooth.

(5) Consider the smooth function $det : GL(n, \mathbb{R}) \to \mathbb{R}$.

(a) Using matrix entries A_{ij} as global coordinates on $GL(n, \mathbb{R})$, show that the partial derivatives of the determinant map are given by:

$$\frac{\partial}{\partial A_{ij}} det(A) = (det\, A)(A^{-1})_{ji}$$

(b) Conclude that the differential of the determinant function is

$$d(det)_A(B) = (det\, A) tr(A^{-1}B)$$

for $A \in GL(n, \mathbb{R})$ and $B \in T_A GL(n, \mathbb{R}) \cong M(n, \mathbb{R})$, where $tr(A) = \Sigma_i A_{ii}$ is the trace of the matrix A.