SOLUTIONS

(1) Over $\mathbb{R}$, consider the open sets $U = (0,1)$, $V = (\frac{1}{2}, \frac{3}{2})$, $W = (\frac{1}{8}, \frac{1}{4}) \cup (\frac{9}{8}, \frac{5}{4})$. Let I_U, I_V be the identity charts on U and V respectively, and $f(x) = \sqrt{x - \frac{3}{16}}$ be a chart on W. Then $\frac{3}{16} \in U \cap W$ and I_U is not related to f.

(2) Infinitely(uncountable) many. For each $p \in (0,1)$, set $\phi_p(x) = x^p$. Since for $p \neq q$, $(\phi_p \circ \phi_q^{-1})(x) = x^{-qp}$, ϕ_p is not related to ϕ_q. Moreover, since $(0,1)$ is uncountable, the collection $\{\phi_p\}_p$ is uncountable.

(3) Let $U = M - \{[0,1]\}$ and $V = M - \{[0,-1]\}$, then $U \cup V = M$. Take the projection on the first coordinate $\pi_1(x,y) = x$, as local chart. It follows that M is locally euclidean. It's also second countable because the projection map $\pi : X \to M$ is open. It's not Hausdorff because one can not separate the point $[0,1]$ from $[0,-1]$ on M, since any neighborhood of $[0,1]$ intersects a neighborhood of $[0,-1]$.

(4) Let $\sigma(x_1, \dots, x_{n+1}) = \frac{1}{1-x_{n+1}}(x_1, \dots, x_n)$ be the stereographic projection chart on $\mathbb{S}^n - N$, with inverse $\sigma^{-1}(u_1, \dots, u_n) = \frac{1}{1+|u|^2}(2u_1, \dots, 2u_n, |u|^2 - 1)$, and let $\bar{\sigma}(x) = \sigma(-x)$ be the chart for the north pole N.

(a) The local expression for A on $\mathbb{S}^n - N$ is

$$(\bar{\sigma} \circ A \circ \sigma^{-1})(u_1, \dots, u_n) = (u_1, \dots, u_n)$$

which is clearly smooth. The same reasoning applies to $\mathbb{S}^n - S$.

(b) Note that as function in $\mathbb{R}^4$, F takes the following form:

$$F(x,y,z,w) = (2(xz + yw), 2(-xw + zy), x^2 + y^2 - z^2 - w^2) \tag{0.1}$$

One the local expression for F is:

$$\begin{aligned}
&(\bar{\sigma} \circ F \circ \sigma^{-1})(u_1, u_2, u_3) = \\
&= \bar{\sigma} \circ F(\frac{1}{1+|u|^2}(2u_1, 2u_2, 2u_3, |u|^2 - 1)) \\
&= \bar{\sigma}(2(\frac{2u_1}{1+|u|^2}\frac{2u_3}{1+|u|^2} + \frac{2u_2}{1+|u|^2}\frac{|u|^2-1}{1+|u|^2}) \\
&, 2(\frac{-2u_1}{1+|u|^2}\frac{|u|^2-1}{1+|u|^2} + \frac{2u_3}{1+|u|^2}\frac{2u_2}{1+|u|^2}) \\
&, \frac{4u_1^2}{(1+|u|^2)^2} + \frac{4u_2^2}{(1+|u|^2)^2} - \frac{4u_3^2}{(1+|u|^2)^2} - \frac{(|u|^2-1)^2}{(1+|u|^2)^2}) \\
&= \frac{(1+|u|^2)^2}{8u_1^2 + 8u_2^2}(-2(\frac{2u_1}{1+|u|^2}\frac{2u_3}{1+|u|^2} + \frac{2u_2}{1+|u|^2}\frac{|u|^2-1}{1+|u|^2}) \\
&, -2(\frac{-2u_1}{1+|u|^2}\frac{|u|^2-1}{1+|u|^2} + \frac{2u_3}{1+|u|^2}\frac{2u_2}{1+|u|^2})) \\
&= \frac{1}{2(u_1^2 + u_2^2)}(-2u_1u_3 - u_2(|u|^2 - 1), u_1(|u|^2 - 1) + 2u_3u_2)
\end{aligned} \tag{0.2}$$

which is smooth when $u_1, u_2 \neq 0$, if we use $(\sigma \circ F \circ \sigma^{-1})$ instead, the denominator in the last equation will be

$$(1 + |u|^2)^2 - 4u_1^2 - 4u_2^2 + 4u_3^2 + (|u|^2 - 1)^2$$

which is $2|u|^4 + 2 - 4u_1^2 - 4u_2^2 + 4u_3^2$. If $u_1 = u_2 = 0$, the last expression is $2u_3^4 + 2 + 4u_3^2$, which is never zero.

(5) Let NC be the the collection of all vectors normal to C. A point in NC is of the form (p, v) where p is in C and v is normal. For every $(p, v) \in NC$, let (x, U) be a chart in C around p. If $v = v_1 e_1 + v_2 e_2$, where $\{e_1, e_2\}$ is a basis for the normal plane at p, the map $\phi(p, v) = (x(p), v_1, v_2)$ satisfies all the conditions of lemma 1.14 of Lee's book, hence define a smooth structure on NC.