SOLUTIONS

(1) Let (p,v) and (q,w) points in TM. If $p \neq q$, let $(x,U),(y,V)$ be charts around p and q respectively. Since M is Hausdorff by definition, we can assume $U \cap V = \emptyset$. Then $TU \cap TV = \emptyset$, since $TU = \{(p,v) : p \in U, v \in T_pM\}$ and $U \cap V = \emptyset$. Now suppose $p = q$, let (x,U) be a chart around p. Note that T_pM is a vector space, in particular is Hausdorff. Let A and B be two open sets containing v and w respectively. Then $(U \times A) \cap (U \times B) = \emptyset$, moreover $(U \times A)$ and $(U \times B)$ are open sets of TM.

(2) (a) By using the definition of the charts on TM, we have:

$$(\bar{\psi} \circ \bar{\phi}^{-1})(x_1, \ldots, x_n, a_1, \ldots, a_n) = \bar{\psi}(x^{-1}(x_1, \ldots, x_n), \Sigma a_i \frac{\partial}{\partial x_i})$$

the latter is equal to:

$$(y \circ x^{-1}(x_1, \ldots, x_n), \Sigma a_i \frac{\partial y_1}{\partial x_i}, \ldots, \Sigma a_i \frac{\partial y_n}{\partial x_i})$$

Hence the Jacobian matrix is:

$$\begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} & 0 & \cdots & 0 \\ \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_2}{\partial x_n} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ 0 & \cdots & 0 & \frac{\partial y_2}{\partial x_1} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix} \tag{0.1}$$

(b) Follows directly from 0.1.

(3) Fix $p \in M$, (x,U) a chart at p, and let $S = \{q \in M : f(q) = f(p)\}$. Since $F_* = \Sigma \frac{\partial F}{\partial x_i}$, $F_* = 0 \to \frac{\partial F}{\partial x_i} = 0$ for each i, therefore the local expression of F, say $\hat{F}$, which is a real valued function, has all partial derivatives zero, hence is constant on $x(U)$. It follows that F is constants in a neighborhood of p, so the set S is open, moreover it's also closed since F (being smooth) is continuous. By the connectness of M, $S = M$.

(4) (This problem illustrates the famous 'Hairy ball theorem'(`https://en.wikipedia.org/wiki/Hairy_ball_theorem`)) This is one of those "guess me" question. You will probably see these types of exercises a lot in your courses. The best approach to such type of question is to start by trying simple things first. For example, the vector field $\frac{\partial}{\partial x_i}$, where (x_1, x_2, x_3) are stereographic coordinates defined on the sphere minus the north pole. Let's choose $\frac{\partial}{\partial x_1}$. Then it is a smooth non zero vector field defined over $\mathbb{S}^2 - N$. If (y_1, y_2, y_3) are the coordinates on $\mathbb{S}^2 - S$, let's see what is the expression

for $\frac{\partial}{\partial x_1}$ in terms of the y_i's. For every $p \in \mathbb{S}^2 - \{N, S\}$:

$$\frac{\partial}{\partial x_1} = \frac{\partial y_1}{\partial x_1}\frac{\partial}{\partial y_1} + \frac{\partial y_2}{\partial x_1}\frac{\partial}{\partial y_2}$$

Recal that:

$$(y \circ x^{-1})(x_1, x_2) = y(\frac{2x_1}{|x|^2+1}, \frac{2x_2}{|x|^2+1}, \frac{|x|^2-1}{|x|^2+1}) = (-\frac{x_1}{|x|^2}, -\frac{x_2}{|x|^2})$$

Therefore

$$\frac{\partial}{\partial x_1} = \frac{x_1^2 - x_2^2}{|x|^4}\frac{\partial}{\partial y_1} + \frac{2x_1x_2}{|x|^4}\frac{\partial}{\partial y_2}$$
$$\frac{\partial}{\partial x_1} = (y_1^2 - y_2^2)\frac{\partial}{\partial y_1} + 2y_1y_2\frac{\partial}{\partial y_2}$$

Now note that $(y_1^2 - y_2^2)\frac{\partial}{\partial y_1} + 2y_1y_2\frac{\partial}{\partial y_2}$ vanishes at the north pole (N=(0,0) in y-coordinates). So indeed, $\frac{\partial}{\partial x_1}$ is what we want! We just define a global vector field on the sphere which is $\frac{\partial}{\partial x_1}$ on $\mathbb{S}^2 - N$ and $(y_1^2 - y_2^2)\frac{\partial}{\partial y_1} + 2y_1y_2\frac{\partial}{\partial y_2}$ on $\mathbb{S}^2 - S$. It is smooth and well defined on the charts' intersection.

(5) Please take a moment to read Staples' proof of a variant of this question using characteristic classes: `http://www.ams.org/journals/proc/1967-018-03/S0002-9939-1967-0219082-6/S0002-9939-1967-0219082-6.pdf`

We first show that $\mathbb{S}^1$ is parallelizable, which is to say that there is smooth global frame. Since $T_p\mathbb{S}^1$ is 1-dimensional for every p, what we have to show is that there is a smooth nowhere vanishing vector field defined on the whole $\mathbb{S}^1$. By adapting the proof from the previous question, we see that $\frac{d}{dx}$ vanishes, hence we need to try another one in this case. Well, the other obvious one is $\frac{d}{dy}$, where y is coordinate on $\mathbb{S}^1 - S$, but by symmetry $\frac{d}{dy}$ would vanish as well. So we consider the next logical candidate $\frac{d}{d\theta}$, where θ is the polar coordinate on $\mathbb{S}^1$. More precisely, we have the chart $(f, \mathbb{S}^1 - \{x \le 0\})$, where $f(x,y) = arctan(\frac{y}{x})$ and $f^{-1}(\theta) = (cos\theta, sin\theta)$. Then:

$$\frac{d}{d\theta} = \frac{dx}{d\theta}\frac{d}{dx}$$

but $(x \circ f^{-1})(\theta) = \frac{cos\theta}{1-sin\theta}$, therefore $\frac{dx}{d\theta} = \frac{1}{1-sin\theta}$, so

$$\frac{d}{d\theta} = \frac{1}{1-sin\theta}\frac{d}{dx} = \frac{x^2+1}{2}\frac{d}{dx}$$

and voila! Now we have a strong candidate for a global frame. Let's see it's expression on the y-coordinate. Recall that $\frac{d}{dx} = \frac{dy}{dx}\frac{d}{dy}$, and $y(x) = \frac{-1}{x}$. We conclude that:

$$\frac{x^2+1}{2}\frac{d}{dx} = \frac{1+y^2}{2}\frac{d}{dy}$$

Thus, the nowhere vanishing vector field $\frac{x^2+1}{2}\frac{d}{dx}$ defines a global frame for $\mathbb{S}^1$. Now we claim that any finite product of $\mathbb{S}^1$ is also parallelizable. Indeed, the global frame (in stereographic coordinates) is given by the n smooth vector fields $0 \oplus \cdots \oplus 0 \oplus \frac{x_i^2+1}{2}\frac{d}{dx_i} \oplus 0 \oplus \cdots \oplus 0$, which is $\frac{x_i^2+1}{2}\frac{d}{dx_i}$ on the i-th position and 0 everywhere else. Where we are using the fact that

$T_{(p,q)}(M, N) \cong T_pM \oplus T_qN$, which follows from taking the derivatives of the projection $M \times N \to M$ and $M \times N \to N$.