SOLUTIONS

(1) Suppose that there is a smooth covector field ω with such property, and let $\omega = f_1 dx_1 + f_2 dx_2 + \cdots + f_n dx_n$. For every curve we have:

$$\int_\gamma \omega = \int_a^b |\gamma'(t)| dt$$

In particular, consider the curves $\alpha(t) = (1-t, 0, \ldots, 0)$ and $\beta(t) = (t, 0, \ldots, 0)$ defined on $[0,1]$. Clearly, $L(\alpha) = L(\beta) = 1$. We have:

$$\begin{aligned} \int_\alpha \omega &= \int_0^1 (f_1 \circ \alpha)\alpha_1'(t) dt \\ &= -\int_0^1 (f_1 \circ \alpha) dt \\ &= -\int_0^1 f_1(1-t, 0, \ldots, 0) dt \\ &= -\int_0^1 f_1(t, 0, \ldots, 0) dt \\ &= -\int_\beta \omega \end{aligned} \tag{0.1}$$

Thus $L(\alpha) = -L(\beta)$, a contradiction.

(2) (a) If $X = \nabla f = \sum \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}$. Then $\int_\gamma X \cdot ds = \int_a^b \sum \frac{\partial f(\gamma(t))}{\partial x_i} \gamma_i'(t) = \int_a^b (f \circ \gamma)'(t) dt = f(\gamma(b)) - f(\gamma(a)) = 0$. Now suppose X conservative. We claim that the smooth covector field ω_x is conservative, where $\omega_x(Y) = X(x) \cdot Y$. Indeed, $\int_\gamma \omega = \int_\gamma X \cdot ds$ by construction, in particular $\int_\gamma \omega$ depend only of the endpoints. Hence $\omega = df$, which implies that $X = \nabla f$.

(b) This follows directly from the definition of *curl* X, since for any smooth function $\frac{\partial f}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_j \partial x_i}$.

(c) By (b), we only need to prove the converse. Using the same smooth covector field ω from (a), by Proposition 4.27 in Lee's book, we get that ω is closed, hence exact and the result follows.

(3) Since f is smooth, it is continuous. The image of a compact under a continuous map is compact, so the image of f is a compact of the real line,i.e is contained in a closed interval.(Compact in the real line = Bounded and closed) So f attains a maximal and minimal value at M, if f were a real function we would claim that maximal and minimal have zero differential, but not so fast, we are in a manifold setting now! Well, we can "turn" f into a real variable one, more precisely: Let $p \in M$, such that $f(p)$ is maximal and $(x = x_1, \ldots, x_n)$ a system of coordinates around p, and consider $f \circ x^{-1}$. Then $x(p)$ is a maximal point of the real valued function $f \circ x^{-1}$, and we now from calculus that a maximal point has all partial

derivatives equal 0, which is the same thing of saying that $\frac{\partial(f\circ x^{-1})(x(p))}{x_j} = 0$. But $df_p = \sum \frac{\partial f(p)}{x_i}(dx_i)_p$, but $\frac{\partial f(p)}{x_i} = \frac{\partial(f\circ x^{-1})(x(p))}{x_i} = 0$, therefore $df_p = 0$.

(4) Let $\alpha = ((y = y_1, \ldots, y_n), v = (v_1, \ldots v_n))$ be a system of coordinates at $(F(p), v) \in T^*N$, and $\beta = ((x = x_1, \ldots, x_n), w = (w_1, \ldots w_n))$ a system of coordinates at (p, F^*v). The local expression for F is then $\beta \circ F \circ \alpha^{-1}$. Note that $\alpha^{-1}(a_1, \ldots, a_n, b_1, \ldots, b_n) = (F(x^{-1}(a_1, \ldots, a_n)), \sum b_i dy_i)$. So $F^* \circ \alpha^{-1}(a_1, \ldots, a_n, b_1, \ldots, b_n) = F^*(F(x^{-1}(a_1, \ldots, a_n)), \sum b_i dy_i) = (x^{-1}(a_1, \ldots, a_n), \sum b_i d(y_i \circ F))$. Finally, $\beta \circ F^* \circ \alpha^{-1}(a_1, \ldots, a_n, b_1, \ldots, b_n) = (x \circ x^{-1}(a_1, \ldots, a_n), \sum b_i \frac{\partial F_i}{\partial x_1}, \ldots, \sum b_i \frac{\partial F_i}{\partial x_n})$, the latter expression is smooth since F is.

(5) (a) Using the Laplace's formula we have that:

$$det(A) = \sum_j (-1)^{i+j} A_{ij} M_{ij}$$

where M_{ij} is the determinant of the matrix you get by deleting the i-th and j-th row, and fixing the i-th row. Then $\frac{\partial}{\partial A_{ij}} det(A) = (-1)^{i+j} M_{ij}$. Recall that the cofactor matrix is given by $C_{ij} = (-1)^{i+j} M_{ij}$, and the inverse A^{-1} is just $\frac{1}{det(A)} C^T$. Well, then M_{ij} is just $det(A) A^{-1}_{ji}$.

(b) Let $B = \sum B_{ij} \frac{\partial}{\partial A_{ij}}$. By definition:

$$d(det)_A(B) = \sum_{i,j} \frac{\partial det(A)}{\partial A_{ij}} dA_{ij}(B) = \sum_{i,j} det(A) A^{-1}_{ji} B_{ij} = det(A) \sum_{i,j} A^{-1}_{ji} B_{ij} = det(A) tr(A^{-1}B)$$