SOLUTIONS

- Lee's 7.2

  F is locally defined on the open $U_x = \{[x, y, z] : x \neq 0\}$ by:

$$\hat{F}(y, z) = (1 - y^2, y, z, yz)$$

the Jacobian is then:

$$\begin{bmatrix} -2y & 0 \\ 1 & 0 \\ 0 & 1 \\ z & y \end{bmatrix}$$

which has rank 2, hence F is an immersion on U_x. Similarly, F is also an immersion on U_y, U_z, and hence on $\mathbb{P}^2$. The map is obviously smooth because it's given in terms of polynomials. Being smooth implies that it's continuous, we claim F is injective. Indeed, if $F(x, y, z) = F(a, b, c)$ then:

$$\begin{aligned} (x^2 - y^2, xy, xz, yz) &= (a^2 - b^2, ab, ac, bc) \\ x^2 - y^2 &= a^2 - b^2 \\ xy &= ab \\ xz &= ac \\ yz &= bc \end{aligned} \tag{0.1}$$

There are two solutions $(x, y, z) = (a, b, c)$ or $(x, y, z) = -(a, b, c)$, but since they define the same point in $\mathbb{P}^2$, in fact there's only one solution in $\mathbb{P}^2$, and F is injective. The inverse of F is $F^{-1}(a, b, c, d) = [\sqrt{\frac{bc^2}{cd}}, d\sqrt{\frac{b}{cd}}, \sqrt{\frac{dc^2}{bc}}]$, which is continuous on the image of F, since $cd \neq 0$ and $bc \neq 0$. (because a point $[x, y, z] \in \mathbb{P}^2$ can't have $x = y = z = 0$)

- Lee's 7.8

  Since a unitary matrix preserves norm, it follows that the action of $SU(n)$ on $\mathbb{S}^{2n-1}$ is transitive. Hence $\mathbb{S}^{2n-1}$ is a homogeneous space diffeomorphic to $SU(n)/SU(n-1)$. If $n = 2$, this shows that $\mathbb{S}^3$ is diffeomorphic to $SU(2)$ since $SU(1)$ is the trivial group 1.

- Lee's 7.12

  The idea here is to find a Hausdorff(since the action is proper) quotient space which is not locally Euclidean that is also given in terms of group actions. First examples that comes to mind are "the cross" , ball with a hair , book with three pages. The last one has a easy quotient structure. Let G be subgroup of $O(2)$ generated by the rotation by 120 degrees, denote it by r, i.e $G = \{1, r, r^2\}$. G it's clearly compact, since it's finite. Now we define the action on G on the closed 2-ball B^2 (note that this is a manifold with boundary) by letting $r^k \cdot x = x$ if x is in the interior of the ball and $r^k \cdot x = r^k x$ if $x \in \mathbb{S}^1$. This action is not free, because points in the interior are fixed by the whole group, hence we can't apply the quotient theorem here, in fact what we want is to show that this gives a counter-example when G does not acts freely. Let $X := B/G$, since G is compact X is Hausdorff, a neighborhood of a point $x \in X$ is a book with three pages which can't be homeomorphic to a ball since if we remove the spine of the book we are left with three connected components, whereas a ball minus a line segment has at most 2 components.

- Lee's 7.14

  Matrix groups can been as (topological) subspaces of $\mathbb{R}^N$ for some N. Since compact subspaces of $\mathbb{R}^N$ are the ones which are bounded and closed, one way of deciding if a given matrix group is compact is by checking these conditions. Let $M(n)$ be the set of $n \times n$ matrices, since the determinant function $det : M(n) \to \mathbb{R}$ is continuous, $GL(n,\mathbb{R}) = det^{-1}((-\infty, 0) \cup (0, \infty))$ is open and hence not compact. On the other hand, $SL(n,\mathbb{R})$ is closed because it's $det^{-1}(\{1\})$, in fact $SL(1,\mathbb{R}) = 1$ is compact, but for $n \geq 2$, $SL(n,\mathbb{R})$ is not compact, since it's not bounded.The same reasoning applies to $GL(n,\mathbb{C}), SL(n,\mathbb{C})$, and they're both not compact as weel. Finally, the only compact in the given list are $U(n)$ and $SU(n)$. We claim that $U(n)$ is compact, then it follows that $SU(n)$ is compact, because it's a closed subgroup of $U(n)$. That $U(n)$ is closed follows from the fact that the function $A \to AA^*$ is continuous, since $U(n)$ is the inverse image of I, it is closed. The columns of a matrix in $U(n)$ form a orthonormal basis for $\mathbb{C}^n$, thus $U(n)$ is bounded and hence compact.

- Lee's 7.22

  (a) The action is smooth because the components functions are polynomials. Suppose $n \cdot (x, y) = (x, y)$, then $(x + n, (-1)^n y) = (x, y)$, from $x + n = x$, we see that $n = 0$ and hence the action is free. Let $K \in \mathbb{R}^2$ be compact, we claim that $G_K = \{n \in \mathbb{Z} : (nK) \cap K \neq \emptyset\}$ is compact, i.e finite in this case. Indeed, for $(x, y) \in K$, $x + n$ is in K for only finitely many n, because K is bounded, since it's a compact of $\mathbb{R}^2$.

  (b) For $n \in \mathbb{Z}$, $\pi_1(n \cdot (x, y)) = \pi_1(x + n, (-1)^n y) = x + n$. So the map $\pi : E \to \mathbb{S}^1$, given by $\pi([x, y]) = e^{2\pi i(\pi_1([x,y]))}$ is well defined. Moreover, π is smooth because it is the composition of smooth maps.

  (c) For any $e^{2\pi ix} \in \mathbb{S}^1$, $\pi^{-1}(e^{2\pi ix}) = \{[x, y] \in E : y \in \mathbb{R}\} \cong \mathbb{R}$, which is a 1 dimensional vector space. Let $U = \mathbb{S}^1 - \{1\}$, then $\pi^{-1}(U) = \{[x, y] \in E : x \notin \mathbb{Z}\}$. Define $\phi : \pi^{-1}(U) \to U \times \mathbb{R}$ by $\phi([x, y]) = (e^{2\pi ix'}, y')$, where $[x', y'] = [x, y]$ and $0 \leq x \leq 1$. Notice that we can always make this choice since $e^{2\pi z}$ is periodic, also we had to remove 1 in order for ϕ to be injective. But now we have to cover 1, we proceed similarly by setting $V = \mathbb{S}^1 - \{-1\}$, and defining $\psi : \pi^{-1}(V) \to V \times \mathbb{R}$ by $\psi([x, y]) = (e^{2\pi ix'}, y')$, where $[x', y']$ is the representative with $\frac{-1}{2} \leq x' \leq \frac{1}{2}$. These maps are diffeomorphism because their coordinates are diffeomorphisms (namely the exponential and the identity). The fact that π maps fibers diffeomorphic is obvious.

  (d) Since π is a rank 1 vector bundle (also called a "line" bundle), a smooth global section is just a nowhere vanishing global section. We claim that there is none and thus π is not trivial. Indeed, global sections of π are smooth maps $s : \mathbb{S}^1 \to E$ such that $s(e^{2\pi ix}) = [x, f(x)]$ for some smooth $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + n) = -f(x)$, well but since $\mathbb{R}$ is connected, the image of f contains 0 and hence vanishes somewhere.