SOLUTIONS

- Lee's exercise 8.8

  (a) $F^*(r\sigma+\tau)_p(X_1,\dots,X_n) = (r\sigma+\tau)_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = r\sigma_{F(p)}(F_*(X_1),\dots,F_*(X_n))+\tau_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = rF^*\sigma_p(X_1,\dots,X_n)+F^*\tau_p(X_1,\dots,X_n)$

  (b)$F^*(f\sigma)_p(X_1,\dots,X_n) = (f\sigma)_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = f(F(p))\sigma_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = (f\circ F)(p)F^*\sigma_p(X_1,\dots,X_n)$

  (c)$F^*(\sigma\otimes\tau)_p(X_1,\dots,X_n) = (\sigma\otimes\tau)_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = \sigma_{F(p)}(F_*(X_1),\dots,F_*(X_n))\otimes\tau_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = F^*\sigma_p(X_1,\dots,X_n)\otimes F^*\tau_p(X_1,\dots,X_n)$

  (d)$(G\circ F)^*\omega_p(X_1,\dots,X_n) = \omega_{(G\circ F)(p)}((G\circ F)_*(X_1),\dots,(G\circ F)_*(X_n)) = \omega_{G(F(p))}((G_*\circ F_*)(X_1),\dots,(G_*\circ F_*)(X_n)) = (G^*\omega)_{F(p)}(F_*(X_1),\dots,F_*(X_n)) = (F^*(G^*\omega))_p(X_1,\dots,X_n)$

  (e) $Id^*\tau_p(X_1,\dots,X_n) = \tau_{Id(p)}(Id_*(X_1),\dots,Id_*(X_n)) = \tau_p(X_1,\dots,X_n)$
- Lee's exercise 8.9

  $(a)\to(b)$ If $\sigma\in S_k$ is any permutation then we claim that $T(X_{\sigma(1)},\dots,X_{\sigma(k)})$ is $T(X_1,\dots,X_k)$. Indeed, start by exchanging $X_{\sigma(1)}$ with X_1, since T is symmetric the value of T remains the same, if we keep doing this until we reach X_k, the value remains the same.

  $(b)\to(a)$ Just take the permutation that send i to j.

  $(b)\to(c)$ Let $T=\sum T_{i_1i_2\dots i_k}\epsilon^{i_1}\otimes\dots\epsilon^{i_k}$, then $T_{i_1i_2\dots i_k}=T(E_1,\dots,E_k)$. Since $T(E_{\sigma(1)},\dots,E_{\sigma(k)})=T(E_1,\dots,E_k)$, we have that $T_{i_1\dots i_k}=T_{\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)}$.

  $(c)\to(a)$ Take $\sigma=(ij)$, then $T(E_1,\dots,E_i,\dots,E_j,\dots,E_k)=T(E_1,\dots,E_j,\dots,E_i,\dots,E_k)$, since T is k-linear we can extend this property for any $X=\sum E_j$.
- Lee's 8.3

  Let $f: V^*\otimes W\to Hom(V,W)$ be the map $f(T\otimes w)(p)=T(p)w$, since W is a vector space, f is clearly linear. We claim that f is injective, if $f(T\otimes w)=0$ then for every $p\in V$, $T(p)w=0$, if $w\neq 0$ then $T(p)=0$ for every p, i.e $T=0$, hence $f(T\otimes w)=0$ implies $T\otimes w=0\otimes w=T\otimes 0=0$. Since $dim(Hom(V,W))=dim(V)\times dim(W)$, by the rank-nulity theorem f is a bijection. Let $\{v_i\}$ be a basis for V with dual basis $\{v_i^*\}$, we set $f^{-1}(T)=\sum v_i^*\otimes T(v_i)$. We check that f^{-1} is the inverse of f: $f(f^{-1}(T))(p)=f(\sum v_i^*\otimes T(v_i))=\sum f(v_i^*\otimes T(v_i))(p)=\sum v_i^*(p)T(v_i)=T(\sum v_i^*(p)v_i)=T(p)\Rightarrow f(f^{-1}(T))=T$. Conversely, $f^{-1}(f(T\otimes w))=\sum v_i^*\otimes f(T\otimes w)(v_i)=\sum v_i^*\otimes T(v_i)w=(\sum T(v_i)v_i^*)\otimes w=T\otimes w$. Since the tensor product is linear in its entries, f^{-1} is linear. Hence, f is an isomorphism.
- Lee's 8.4

  Evaluating both sides at $(\frac{\partial}{\partial x_{i_1}},\dots,\frac{\partial}{\partial x_{i_k}})$ we get:

$$\sigma_{i_1\dots i_k}=\sum_J\overline{\sigma}_{j_1\dots j_k}d\overline{x}^{j_1}(\frac{\partial}{\partial x_{i_1}})\otimes\dots\otimes d\overline{x}^{j_k}(\frac{\partial}{\partial x_{i_k}})=\sum_J\overline{\sigma}_{j_1\dots j_k}\frac{\partial\overline{x}^{j_1}}{\partial x_{i_1}}\cdots\frac{\partial\overline{x}^{j_k}}{\partial x_{i_k}}$$

- Lee's 8.7

Let $\{\epsilon_1, \ldots, \epsilon_n\}$ be a dual basis for V. Then as a vector space, $\Sigma^k(V)$ is generated by the collection $\epsilon_{i_1} \odot \cdots \odot \epsilon_{i_k}$ with $1 \leq \epsilon_{i_k} \leq \cdots \leq \epsilon_{i_k} \leq n$(repetitions allowed). The number of such elements is then the number of multisets of length k taken from a set of length n. From combinatorics we know that this is $\binom{n+k-1}{k}$. This is often called informally "stars and bars theorem". To see why this is true, we want to count the number of k-arrangements of stars separated by $n-1$ bars, or we want to choose k elements of a set with $k+(n-1)$ elements.