SOLUTIONS

- Tu's 19.5

$$
\begin{aligned}
c^*\omega &= c^*(\frac{-y}{x^2+y^2})d(\cos t) + c^*(\frac{x}{x^2+y^2})d(\sin t) \\
&= \frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t)dt + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t)dt \\
&= dt
\end{aligned} \tag{0.1}
$$

- Tu's 19.9 Let $\omega = \sum \omega_i dx_i$, then $d\omega = \sum d\omega_i \wedge dx_i$. Hence:

$$
\begin{aligned}
d\omega(X,Y) &= \sum_i d\omega_i(X)dx_i(Y) - d\omega_i(Y)dx_i(X) \\
&= \sum_i X(\omega_i)Y(x_i) - Y(\omega_i)X(x_i)
\end{aligned} \tag{0.2}
$$

on the other hand:

(0.3)

$$
\begin{aligned}
X\omega(Y) - Y\omega(X) - \omega([X,Y]) &= X\sum \omega_i Y(x_i) - Y\sum \omega_i X(x_i) - \sum \omega_i [X,Y](x_i) \\
&= \sum X(\omega_i)Y(x_i) + \omega_i XY(x_i) - Y(\omega_i)X(x_i) \\
&- \omega_i YX(x_i) - \omega_i XY(x_i) + \omega_i YX(x_i) \\
&= \sum_i X(\omega_i)Y(x_i) - Y(\omega_i)X(x_i)
\end{aligned}
$$

- Lee's 9.1 Let $\{u_1, \ldots, u_n\}$ be the associated Gram-Schmidt elements. Then $\det(v_1, \ldots, v_n) = \det(u_1, \ldots, u_n)$. Note that the volume of a parallelepiped is defined inductively: it is the length of u_n, times the volume of $span\{u_1, \ldots, u_{n-1}\}$, i.e base times height. This is true for $n = 2$, indeed if $v_1 = (a,b), v_2 = (c,d)$, from plane geometry we get $|v_1||v_2|\sin\theta = |v_1||u_2| = \langle v_1, u_2\rangle = ad - bc$. Suppose it's true for $n-1$, then the det of $[u_1, \ldots, u_n][u_1, \ldots, u_n]^T$ is $[u_n]*[u_n]^T$ times $[u_1, \ldots, u_{n-1}][u_1, \ldots, u_{n-1}]^T$, since u_n is orthogonal to the other u_i, using the induction hypothesis we get this is $|u_n|^2 \det(u_1, \ldots, u_{n-1})^2$, hence $\det(u_1, \ldots, u_n)^2 = |u_n|^2 \det(u_1, \ldots, u_{n-1})^2$, which is the square of the area of the parallelogram by definition.
- Lee's 9.2

  Suppose $e_1 \otimes e_2 \otimes e_3 = a + s$. Note that $(e_1 \otimes e_2 \otimes e_3)(E_1, E_2, E_3) = 1$, where (E_1, E_2, E_3) are the basis elements to which e_i are dual. Now $(e_1 \otimes e_2 \otimes e_3)(E_1, E_2, E_3) = 1 = a(E_2, E_3, E_1) + s(E_2, E_3, E_1) = (e_1 \otimes e_2 \otimes e_3)(E_2, E_3, E_1) = 0$ a contracdiction.
- Lee's 9.6

(a) Let $I : \mathbb{R}^3 \to \mathbb{R}^3$ be the identity $I(x) = x$, then

(0.4)

$$
\begin{aligned}
I^*\Omega = {} & \rho\sin\phi\cos\theta d(\rho\sin\phi\sin\theta) \wedge d(\rho\cos\phi) \\
& + \rho\sin\phi\sin\theta d(\rho\cos\phi) \wedge d(\rho\sin\phi\cos\theta) \\
& + \rho\cos\phi d(\rho\sin\phi\cos\theta) \wedge d(\rho\sin\phi\sin\theta) \\
= {} & \rho\sin\phi\cos\theta(\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta) \wedge (\cos\phi d\rho - \rho\sin\phi d\phi) \\
& + \rho\sin\phi\sin\theta(\cos\phi d\rho - \rho\sin\phi d\phi) \wedge (\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta) \\
& + \rho\cos\phi(\sin\phi\cos\theta d\rho + \rho\cos\phi\cos\theta d\phi - \rho\sin\phi\sin\theta d\theta) \wedge (\sin\phi\sin\theta d\rho + \rho\cos\phi\sin\theta d\phi + \rho\sin\phi\cos\theta d\theta) \\
= {} & \rho^3\sin\phi d\phi \wedge d\theta
\end{aligned}
$$

(b) $d\Omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy = 3dx \wedge dy \wedge dz$
Also:
$d\Omega = 3\rho^2 \sin\phi d\rho \wedge d\phi \wedge d\theta$

They define the same expression because the Jacobian of the change of coordinates is $\rho^2 \sin\phi$.

(c) The restrciction to $\mathbb{S}^2$ is when $\rho = 1$, hence $\iota^*\Omega = \sin\phi d\phi \wedge d\theta$

(d) Note that $\phi \in (0, \pi)$. Since $\sin\phi > 0$ in this interval, $\iota^*\Omega$ is never zero.