SOLUTIONS

- Lee's 10.1

  You can see why this is true be just thinking how you leave and come back to your home! If you leave your house walking forward you can *always* come back to it walking forward again, same thing if you leave it walking backwards. The path described by you is then a closed (oriented) curve, which is a 1-manifold. We can generalize this notion to any 1-manifold. More precisely, let M be a smooth 1-manifold. Since for any smooth n-manifold X, the image of every chart can be assumed to be a ball in $\mathbb{R}^n$, we can assume that M is locally diffeomorphic to 1-ball, i.e open intervals. The idea is to choose an orientation for one interval and then spread it to all the intervals which has non empty intersection with it, note that since the intersection of two intervals is an interval, it's connected so that the determinant doesn't change sign(more on this in the next question). The worse case scenario is when the interval we start with has non-zero intersection on both ends (every interval intersect each other on both ends in this case), i.e a closed smooth curve = a circle, but we've already proved that the circle is orientable, hence all 1-manifolds are orientable.
- Lee's 10.2

  By assumption, M can be covered with only 2 charts, say U and V, with $U \cap V$ connected. Also, recall that the function $sign = \frac{x}{|x|}$ is continuous if $x \neq 0$, in particular $sign \circ det$ is continuous on $GL(n)$, so it sends connected sets to connected,i.e either 1 or -1. Therefore, on $U \cap V$, the determinant function does not change sign, hence if the jacobian is positive it will be so on the whole intersection, same thing if it is negative. If it's negative, we use the chart with "minus first coordinate", so that it will become positive. Hence there is a well-defined orientation on $M = U \cup V$. Since $\mathbb{S}^n$ can be covered by only 2 open sets ($U = \mathbb{S}^n - N, V = \mathbb{S}^n - S$), with $U \cap V$ connected, it follows that $\mathbb{S}^n$ is orientable.
- Lee's 10.4

  Let $\pi : M \to M/\Gamma$ be the quotient projection. Then π_* is an isomorphism (M and M/Γ have same dimension) at every $x \in M$. Given $[x] \in M/\Gamma$, choose x, such that $\pi(x) = [x]$. Using the orientation on M and the isomorphism $(\pi_*)_x : T_xM \cong T_{[x]}M/\Gamma$, we want to orient $T_{[x]}M/\Gamma$ for every $[x]$. We claim that this choice is independent of the representative x if $T_\gamma(x) := \gamma \cdot x$ is orientation preserving. Indeed, choose $y \in [x]$, then $y = \gamma x$, by hypothesis, $(T_\gamma)_{*,x} : T_xM \to T_yM$, is orientation preserving, hence $(\pi_*)_y$ induces the same orientation as $(\pi_*)_x$. This point-wise orientation is continuous because frames on M induces frames on M/Γ, since π_* is an isomorphism at every $x \in M$. Conversely, suppose M/Γ orientable and let $I : M/\Gamma \to M/\Gamma$ be the identity map. Then $I \circ \pi = \pi \circ T_\gamma$, so $\pi_* = (I \circ \pi)_* = (\pi \circ T_\gamma)_* = \pi_* \circ (T_\gamma)_*$. If $(T_\gamma)_*$ reverses orientation and

π_* preserves we get contradiction, same thing if π_* reverses orientation. Hence, $(T_\gamma)_*$ has to preserve orientation.

- Lee's 10.5

  By 10.4, E(or M as a manifold with boundary) is orientable if and only if the $\mathbb{Z}$-action is orientation preserving. The Jacobian of the action is $(-1)^n$, hence if n is odd, the action is orientation reversing, thus E(or M) is not orientable.
- Lee's 10.7

  Let $I = [0, 2\pi]$, and $F : I \times I \to \mathbb{T}^2$ be the map
  $$F(\alpha, \beta) = (\cos\alpha, \sin\alpha, \cos\beta, \sin\beta)$$
  We claim that F is orientation preserving. Note that $F_*(\frac{\partial}{\partial\alpha}) = -\sin\alpha\frac{\partial}{\partial w} + \cos\alpha\frac{\partial}{\partial x}$ and $F_*(\frac{\partial}{\partial\beta}) = -\sin\beta\frac{\partial}{\partial y} + \cos\beta\frac{\partial}{\partial z}$. Recall that the volume form of $\mathbb{S}^1$ is just the restriction of the form $v = xdy - ydx$ defined on $\mathbb{R}^2$. But $v(F_*(\frac{\partial}{\partial\alpha})) = 1$ which is positive, similarly $v(F_*(\frac{\partial}{\partial\beta})) = 1$. Therefore, $[F_*(\frac{\partial}{\partial\alpha}), F_*(\frac{\partial}{\partial\beta})]$ is an oriented basis of the product orientation and hence F preserves orientation. Then
  $$\begin{aligned}\int_{\mathbb{T}^2}\omega &= \int_{I\times I} F^*\omega \\ &= \int_0^{2\pi}\int_0^{2\pi}\cos\alpha\cos\beta(\cos\alpha\cos\beta)d\alpha d\beta \\ &= \int_0^{2\pi}\int_0^{2\pi}\cos^2\alpha\cos^2\beta d\alpha d\beta \\ &= \int_0^{2\pi}\cos^2\alpha d\alpha\int_0^{2\pi}\cos^2\beta d\beta \\ &= (\int_0^{2\pi}\cos^2\alpha d\alpha)^2 \\ &= \pi^2\end{aligned} \tag{0.1}$$