SOLUTIONS

- Lee's 17.1

  $(\omega+da)\wedge(\eta+db) = \omega\wedge\eta+\omega\wedge db+da\wedge\eta+da\wedge db = \omega\wedge\eta+(-1)^k d(\omega\wedge b)+d(a\wedge\eta)+d(a\wedge db)$, hence $[\omega\wedge\eta]=[\omega]\wedge[\eta]$
- Lee's 17.6

  Let U be a open ball centered at x and $V = M\setminus\{x\}$. Note that $H^0(M)\cong H^0(V)\cong\mathbb{R}$, since M,V are connected. If $1<p<n-1$, the Mayer-Vietoris sequence for the pair $\{U,V\}$ translates to

  $$H^{p-1}(U\setminus\{x\})\to H^p(M)\to H^p(U)\oplus H^p(V)\to H^p(U\setminus\{x\})$$

  since $H^p(U)=0$ for $p>0$ and $H^p(U)=H^p(\mathbb{S}^{n-1})=0$ if $1<p<n-1$, we get that the above sequence is just

  $$0\to H^p(M)\to H^p(V)\to 0$$

  hence $H^p(M)\cong H^p(V)$.

  When $p=n-1$ and M compact orientable, we have

  $$0\to H^{n-1}(M)\to H^{n-1}(V)\to\mathbb{R}\to\mathbb{R}\to 0$$

  by counting dimensions we get that $H^{n-1}(M)\cong H^{n-1}(V)$

  When $p=0,1$ the sequence reads:

  $$0\to\mathbb{R}\to\mathbb{R}\oplus\mathbb{R}\to\mathbb{R}\to H^1(M)\to H^1(V)\to 0$$

  by the same reasoning above, $H^1(M)\cong H^1(V)$.
- Lee's 17.7

  Using the notation of Problem 9-12, we have that

  $$H^p(\widetilde{M}_i)=H^p(M_i\setminus\{p_i\}),\ i=1,2$$

  moreover, by the previous problem if $0<p<n-1$, then $H^p(M_i\setminus\{p_i\})\cong H^p(M_i)$. The Mayer-Vietoris sequence for the pair $\widetilde{M}_1,\widetilde{M}_2$ reads:

  $$H^{p-1}((-1,1)\times\mathbb{S}^{n-1})\to H^p(M_1\# M_2)\to H^p(M_1)\oplus H^p(M_2)\to H^p((-1,1)\times\mathbb{S}^{n-1})$$

  Notice that $(-1,1)\times\mathbb{S}^{n-1}$ has the same homotopy type of S^{n-1}, hence $H^p((-1,1)\times\mathbb{S}^{n-1})\cong H^p(\mathbb{S}^{n-1})$. Therefore, for $0<p<n-1$, we have $0\to H^p(M_1\# M_2)\to H^p(M_1)\oplus H^p(M_2)\to 0$. When M_i are compact and orientable we can use again the previous problem to get:

  $$0\to H^{n-1}(M_1\# M_2)\to H^{n-1}(M_1)\oplus H^{n-1}(M_2)\to\mathbb{R}\to\mathbb{R}\to\mathbb{R}\oplus\mathbb{R}\to 0$$

  By counting dimension again we conclude that $H^{n-1}(M_1\# M_2)\cong H^{n-1}(M_1)\oplus H^{n-1}(M_2)$
- Lee's 17.8

  (a) The correspondence is just $\omega\to[\omega]$, for every orientation form ω. If $[\omega]$ and $[\eta]$ give the same vector space orientation, then $[\omega]=c[\eta]$ for some positive c, in particular $\omega=c\eta$ and ω and η give the same orientation.

(b) Suppose F^* preserves orientation, then $F^*[\omega_N] = [\omega_M]$, which implies that $F^*(\omega_N + d\eta) = \omega_M + d\tau$. Taking $\tau = \eta = 0$ we have in particular that $F^*\omega_N = \omega_M$, and hence for any oriented form $\alpha = c\omega_N$, $F^*\alpha$ is oriented, i.e F preserves orientation. Conversely, suppose F preserves orientation, then F^* takes oriented forms to oriented forms. We have $F^*(\omega_N + d\eta) = F^*(\omega_N) + F^*(d\eta) = \omega_M + d(F^*\eta)$, thus $F^*[\omega_N] = [\omega_M]$.