Exam II

Choose (only) 5 questions:

1. Describe in your own words what does it mean to say that $\mathbb{R}$ is a complete ordered field.

2. Let $X = \{x \in \mathbb{Q}; x^2 < 3\}$. Find $\sup X \in \mathbb{R}$. Explain.

3. Let P be the set of positive elements in a ordered field K. Consider the function $f : P \to P$ given by $f(x) = x^2$. Show that $f(x)$ is increasing, i.e. $x < y \Rightarrow f(x) < f(y)$.

4. A sequence x_n is periodic if there is $p \in \mathbb{N}$ such that $x_{n+p} = x_n$ for every $n \in \mathbb{N}$. Show that every convergent periodic sequence is constant.

5. Give an example of a sequence x_n such that the set of all accumulation points of x_n is $\{-1, 0, 1\}$.

6. Find the set of all accumulation points of the sequence x_n defined by $x_{2n} = \frac{1}{n}$ and $x_{2n-1} = n$.

7. Show that $\forall p \in \mathbb{N}$ we have
$$\lim \sqrt[n+p]{n} = 1$$
Hint: You may use the fact that $\lim \sqrt[n]{n} = 1$.

8. Define a sequence inductively by $x_1 = \sqrt{2}$ and
$$x_{n+1} = \sqrt{2 + x_n}$$
Show that x_n is convergent and find its limit. You may assume (the nontrivial fact) that x_n is bounded.

9. If $\lim x_n = +\infty$, find
$$\lim \left[\sqrt{\log(x_n + 1)} - \sqrt{\log(x_n)}\right]$$