Exam II

Choose (only) 5 questions:

1. Describe in your own words what does it mean to say that $\mathbb{R}$ is a complete ordered field.

   Solution. It means that $\mathbb{R}$ is a set together with two operations $+$ and $\times$, such that these operations satisfy the fields axioms. Moreover, the set $P = \{x \in \mathbb{R}; x > 0\}$ gives the field $\mathbb{R}$ an order. Lastly, $\mathbb{R}$ is complete in the sense that every nonempty bounded set has a supremum, equivalently, every Cauchy sequence of real numbers converge to a real number. □

2. Let $X = \{x \in \mathbb{Q}; x^2 < 3\}$. Find $\sup X \in \mathbb{R}$. Explain.

   Solution. We claim $\sup X = \sqrt{3}$. Indeed, $\sqrt{3}$ is obviously an upper bound. Take $a \in \mathbb{R}$, such that $a < \sqrt{3}$. Then $a^2 < 3$ implies $a \in X$, and we can always find a rational $r \in X$ such that $a < r$, so a is not an upper bound and we conclude that $\sqrt{3}$ is the least upper bound. □

3. Let P be the set of positive elements in a ordered field K. Consider the function $f : P \to P$ given by $f(x) = x^2$. Show that $f(x)$ is increasing, i.e. $x < y \Rightarrow f(x) < f(y)$.

   Solution. Suppose $x < y \in P$, say $y = x + a$, with $a \in P$. Then $y^2 = (x + a)^2 = x^2 + 2xa + a^2$, hence $y^2 > x^2$ since $2xa + a^2 \in P$. □

4. A sequence x_n is periodic if there is $p \in \mathbb{N}$ such that $x_{n+p} = x_n$ for every $n \in \mathbb{N}$. Show that every convergent periodic sequence is constant.

   Solution. If x_n is not constant, there is at least one n_0 such that $x_{n_0} \neq x_{n_0+1}$. Then the constant subsequences x_{n_0+p} and x_{n_0+1+p} converge to different numbers, so x_n itself can't be convergent. This is the contra-positive of the problem's statement. □

5. Give an example of a sequence x_n such that the set of all accumulation points of x_n is $\{-1, 0, 1\}$.

   Solution. $x_n = \cos(\frac{n\pi}{2})$, i.e. $\{0, -1, 0, 1, \ldots\}$ □

6. Find the set of all accumulation points of the sequence x_n defined by $x_{2n} = \frac{1}{n}$ and $x_{2n-1} = n$.

   Solution. x_{2n-1} is unbounded and increasing so any subsequence of it also has this properties and hence diverges. On the other hand, $\lim x_{2n} = 0$ and any subsequence of it, if convergent, also converges to 0. Therefore, 0 is the only accumulation point of x_n. □

7. Show that $\forall p \in \mathbb{N}$ we have

$$\lim \sqrt[n+p]{n} = 1$$

Hint: You may use the fact that $\lim \sqrt[n]{n} = 1$.

Solution. Notice that $1 \leq \sqrt[n+p]{n} \leq \sqrt[n]{n}$, the result follows by the Squeeze thereom. □

8. Define a sequence inductively by $x_1 = \sqrt{2}$ and

$$x_{n+1} = \sqrt{2 + x_n}$$

Show that x_n is convergent and find its limit. You may assume (the nontrivial fact) that x_n is bounded.

Solution. Since x_n is monotone and bounded, it converges, say to $a \in \mathbb{R}$. Taking the limit we obtain

$$a^2 = 2 + a$$

Hence, $a = -1$ or $a = 2$, since $x_n \geq 0 \Rightarrow \lim x_n \geq 0$, we must have $a = 2$. □

9. If $\lim x_n = +\infty$, find

$$\lim \left[\sqrt{\log(x_n + 1)} - \sqrt{\log(x_n)}\right]$$

Solution.

$$\begin{aligned}
\lim \left[\sqrt{\log(x_n + 1)} - \sqrt{\log(x_n)}\right] &= \lim \frac{\log(x_n + 1) - \log(x_n)}{\left[\sqrt{\log(x_n + 1)} + \sqrt{\log(x_n)}\right]} \\
&= \lim \frac{\log(1 + \frac{1}{x_n})}{\left[\sqrt{\log(x_n + 1)} + \sqrt{\log(x_n)}\right]} \\
&= \frac{\lim \log(1 + \frac{1}{x_n})}{\lim \left[\sqrt{\log(x_n + 1)} + \sqrt{\log(x_n)}\right]} \\
&= 0
\end{aligned}$$

□