Radially symmetric solutions to a Lane-Emden type system

Abstract

The existence of radially symmetric solutions is discussed for a Lane-Emden type system. This answer a question posed by da Silva and do O (2024). We also comment on the inhomogeneous version of the same system and discuss some open questions.

Keywords: Nonlinear elliptic equations, Wolff potentials, fractional laplacian, radially symmetric solutions

MSC Classification: 35J70 , 45G15 , 35B09 , 35R11

1 Introduction

In this manuscript we analyze the existence of solutions to the system

$$\begin{cases} (-\Delta)^{\alpha} u = \sigma\, v^{q_1}, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ (-\Delta)^{\alpha} v = \sigma\, u^{q_2}, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x|\to\infty} u(x) = 0, & \liminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1}$$

when $\sigma \in M^+(\mathbb{R}^n)$ is radially symmetric satisfying $\mathbf{I}_{2\alpha}\sigma(x) \not\equiv \infty$ for almost every $x \in \mathbb{R}^n$, or equivalently,

$$\int_1^\infty \left(\frac{\sigma(B(0,t))}{t^{n-2\alpha}} \right) \frac{\mathrm{d}t}{t} < \infty, \tag{2}$$

with $0 < \alpha < n/2$ and $q_1,\ q_2 \in (0,1)$.

A solution (u,v) to (1) is understood in the sense

$$\begin{cases} u(x) = \mathbf{I}_{2\alpha}(v^{q_1}\mathrm{d}\sigma)(x), & x \in \mathbb{R}^n, \\ v(x) = \mathbf{I}_{2\alpha}(u^{q_2}\mathrm{d}\sigma)(x), & x \in \mathbb{R}^n, \end{cases} \tag{3}$$

where $\mathbf{I}_{2\alpha}$ is the Riesz Potential defined by

$$\mathbf{I}_{\alpha}\sigma(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-\alpha}}, \quad x \in \mathbb{R}^n. \tag{4}$$

This type of problem gained attention after the publication of the seminal paper Brezis and Kamin (1992), where the authors gave necessary and sufficient conditions for existence and uniqueness of solutions of the single semilinear elliptic problem

$$-\Delta u = \sigma(x)u^q$$

in $\mathbb{R}^n (n \geq 3)$ with $0 < q < 1,\ 0 \not\equiv \sigma \geq 0$ and $\sigma \in L^\infty_{\text{loc}}(\mathbb{R})$. They proved that a bounded solution exists if and and only if $\mathbf{I}_2\sigma(x) \in L^\infty(\mathbb{R})$. Additionally, the following pointwise bound holds

$$c^{-1} (\mathbf{I}_2\sigma)^{\frac{1}{1-q}} \leq u \leq c\,(\mathbf{I}_2\sigma) \tag{5}$$

where $c > 0$ is a constant independent of u.

Other works followed, and interesting results were published generalizing Brezis and Kamin (1992) to a wider class of problems. For example, in Kilpeläinen and Malý (1992); Kilpeläinen and Malý (1994) the authors generalize the Brezis-Kamin estimates to the quasilinear problem

$$-\Delta_p u = \sigma,$$

obtaining the estimates

$$c^{-1}\mathbf{W}_{1,p}\sigma(x) \leq u(x) \leq c\,\mathbf{W}_{1,p}\sigma(x), \quad x \in \mathbb{R}^n, \tag{6}$$

where $\mathbf{W}_{\alpha,p}\sigma(x)$ is the Wolff potential of σ, introduced in Hedberg and Wolff (1983) and defined by

$$\mathbf{W}_{\alpha,p}\sigma(x) = \int_0^\infty \left(\frac{\sigma(B(x,t))}{t^{n-\alpha p}}\right)^{\frac{1}{p-1}} \frac{\mathrm{d}t}{t}, \quad x \in \mathbb{R}^n, \tag{7}$$

for $\sigma \in M^+(\mathbb{R}^n),\ 1 < p < \infty,\ 0 < \alpha < n/p$ and $B(x,t)$ is the open ball of radius $t > 0$ centered at x. See the wonderful textbooks Adams and Hedberg (2010); Heinonen et al. (2006) for more on nonlinear potentials.

Remark 1. *Notice that*

$$\mathbf{I}_{2\alpha}\sigma(x) = \int_{\mathbb{R}^n} \frac{\mathrm{d}\sigma(y)}{|x-y|^{n-2\alpha}} = (n-2\alpha)\int_0^\infty \frac{\sigma(B(x,t))}{t^{n-2\alpha}}\frac{\mathrm{d}t}{t} = (n-2\alpha)\mathbf{W}_{\alpha,2}\sigma(x),$$

so the Wolff potential coincides (up to a constant) with the Riesz potential when $p = 2$.

In Cao and Verbitsky (2016) the authors analyze the equivalent Brezis-Kamin problem for the p-Laplacian

$$-\Delta_p u = \sigma\, u^q \quad \text{in } \mathbb{R}^n,$$

and obtain (not necessary bounded) solutions satisfying

$$c^{-1} (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}} \leq u \leq c\left(\mathbf{W}_{1,p}\sigma + (\mathbf{W}_{1,p}\sigma)^{\frac{p-1}{p-1-q}}\right) \tag{8}$$

In da Silva and do O (2024), the authors generalize these ideas to systems of the form (1) and obtain similar estimates. More precisely they proved that under certain conditions there exists a solution pair (u, v) to (1) satisfying the following conditions

$$\begin{aligned} c^{-1}\left(\mathbf{I}_{2\alpha}\sigma\right)^{\gamma_1} \leq u \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}\right), \\ c^{-1}\left(\mathbf{I}_{2\alpha}\sigma\right)^{\gamma_2} \leq v \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right), \end{aligned} \tag{9}$$

where

$$\gamma_1 = \frac{1+q_1}{1-q_1q_2},\ \gamma_2 = \frac{1+q_2}{1-q_1q_2}.$$

Remark 2. *Based on the assumption (2), they also show that all nontrivial solutions to (1) satisfy the lower bounds in (9).*

In the same paper they proposed a question of whether or not a criteria could be found in the case σ is radially symmetric, generalizing (Cao and Verbitsky, 2016, Prop. 5.2) to the case of systems. Our first goal is to answer this question positively and prove the following theorem

Theorem 1. *Let $\sigma \in M^+(\mathbb{R}^n)$ be radially symmetric. Then there is a solution (u, v) to system (1) satisfying (9) if and only if*

$$\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty \ \textit{and} \ \limsup_{|x|\to 0} \frac{\frac{1}{|x|^{(n-2\alpha)\frac{1}{\gamma_i}}} \int_{B(0,|x|)} \frac{d\sigma(y)}{|y|^{r_i(n-2\alpha)}}}{\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}}} < \infty, i = 1, 2. \tag{10}$$

where $r_1 = 1 - \frac{1}{\gamma_1}$ and $r_2 = 1 - \frac{1}{\gamma_2}$.

Our second result consists of analyzing the existence of solutions to a generalization of system (1):

$$\begin{cases} (-\Delta)^\alpha u = \sigma\, v^{q_1} + \mu_1 \\ (-\Delta)^\alpha v = \sigma\, u^{q_2}, +\mu_2 \\ \liminf\limits_{|x|\to\infty} u(x) = 0, \quad \liminf\limits_{|x|\to\infty} v(x) = 0. \end{cases} \tag{11}$$

where $\sigma, \mu_1, \mu_2 \in M^+$ are not necessarily radially symmetric but σ satisfies the condition

$$\sigma(E) \leq C_\sigma\, \mathrm{cap}_{\alpha,2}(E) \quad \text{for all compact sets } E \subset \mathbb{R}^n. \tag{12}$$

We understand system (11) as

$$\begin{cases} u = \mathbf{I}_{2\alpha}\left(v^{q_1}\mathrm{d}\sigma + d\mu_1\right), \\ v = \mathbf{I}_{2\alpha}\left(u^{q_2}\mathrm{d}\sigma + d\mu_2\right), \end{cases} \tag{13}$$

We'll show that if $\mathbf{I}_{2\alpha}\mu_i(x) \leq \mathbf{I}_{2\alpha}\sigma(x)$ then we still have existence, more precisely we have:

Theorem 2. *Let $\sigma, \mu_1, \mu_2 \in M^+(\mathbb{R}^n)$ not necessarily radially symmetric where σ satisfies (12). Suppose that for $i = 1, 2$, there is a constant $C > 0$ such that*

$$\mathbf{I}_{2\alpha}\mu_i(x) \leq C\, \mathbf{I}_{2\alpha}\sigma(x)$$

Then there is a solution (u,v) to system (11) satisfying (9).

Notations and definitions

We assume $\Omega \subseteq \mathbb{R}^n$ is a domain. We denote by $M^+(\Omega)$ the space of all nonnegative locally finite Borel measures on Ω and $\sigma(E) = \int_E d\sigma$ the σ-measure of a measurable set $E \subseteq \Omega$. The letter c will always denote a positive constant which may vary from line to line.

Organization of the paper

In Sect. 2, we present a criteria for existence of solutions to (1) when σ is radial, in Sect. 3 we give the proof of theorem 1 and in Sect. 4 we prove theorem 2.

2 Conditions for existence

In this section we give necessary and sufficient conditions for existence of solutions to the system (1) in case $\sigma \in M^+(\mathbb{R}^n)$ is radially symmetric.

The following result can be considered as a generalization of (Cao and Verbitsky, 2016, Prop. 5.1).

Theorem 3. *Let $\sigma \in M^+(\mathbb{R}^n)$ be radially symmetric. Then there exists a nontrivial solution pair (u,v) to (1) if and only if*

$$\int_{|y|<1} \frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}} < \infty, \int_{|y|<1} \frac{d\sigma(y)}{|y|^{(n-2\alpha)r_2}} < \infty \text{ and } \int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty, \tag{14}$$

where $r_i = 1 - \frac{1}{\gamma_i}$. Moreover, we have

$$\begin{aligned} u(x) &\approx \left(\frac{1}{|x|^{n-2\alpha}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}} \right)^{\gamma_1} + \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\gamma_1} \right) \\ v(y) &\approx \left(\frac{1}{|y|^{n-2\alpha}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{\gamma_2} + \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\gamma_2} \right). \end{aligned} \tag{15}$$

Proof. Recall that if σ is radial then

$$\mathbf{I}_{2\alpha}\sigma(x) \approx \frac{\sigma(B(0,|x|))}{|x|^{n-2\alpha}} + \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}},$$

where we omit the first term when $x = 0$. Moreover, we can assume u, v are radial since $\mathbf{I}_{2\alpha}\sigma(x)$ is radial in this case.

($\Rightarrow$) Suppose (u,v) is a solution to (1). Then according to remark 2

$$\begin{aligned} c\,(\mathbf{I}_{2\alpha}\sigma)^{\gamma_1} &\leq u \\ c\,(\mathbf{I}_{2\alpha}\sigma)^{\gamma_2} &\leq v. \end{aligned} \tag{16}$$

In particular, we have

$$c\left(\int_{|y|\geq x}\frac{d\sigma(y)}{|y|^{n-2\alpha}}\right)^{\gamma_1}\leq u, \tag{17}$$

but since u is finite, this implies $\int_{|y|\geq 1}\frac{d\sigma(y)}{|y|^{n-2\alpha}}<\infty$. Likewise,

$$c\left(\frac{1}{|x|^{(n-2\alpha)}}\int_{|y|\leq x}\frac{|y|^{(n-2\alpha)r_1}d\sigma(y)}{|y|^{(n-2\alpha)r_1}}\right)^{\gamma_1}\leq u, \tag{18}$$

For any $\delta>0$ such that $\delta<|y|$ we have

$$c\left(\frac{\delta^{(n-2\alpha)r_1}}{|x|^{(n-2\alpha)}}\int_{\delta<|y|\leq x}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}}\right)^{\gamma_1}\leq\left(\frac{1}{|x|^{(n-2\alpha)}}\int_{|y|\leq x}\frac{|y|^{(n-2\alpha)r_1}d\sigma(y)}{|y|^{(n-2\alpha)r_1}}\right)^{\gamma_1}\leq u, \tag{19}$$

By symmetry, the exact same argument works with v, r_2, γ_2 instead of u, r_1, γ_1.

($\Leftarrow$) Conversely, suppose (14) holds. Notice that

$$\gamma_1=q_1\gamma_2+1,\ \gamma_2=q_2\gamma_1+1,$$

According to (da Silva and do O, 2024, Proof of thm. 1), there is a $\lambda_1>0$ sufficiently small such that

$$(\underline{u},\underline{v})=\left(\lambda_1(\mathbf{I}_{2\alpha}\sigma)^{\gamma_1},\lambda_1(\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right)$$

is a subsolution to (1). So it's enough to find supersolution $(\overline{u},\overline{v})$ such that $\overline{u}\geq\underline{u}$ and $\overline{v}\geq\underline{v}$. Set

$$\overline{u}(x)=\lambda\left(\frac{1}{|x|^{n-2\alpha}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}}\right)^{\gamma_1}+\left(\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2\alpha}}\right)^{\gamma_1}\right)$$
$$\overline{v}(y)=\lambda\left(\frac{1}{|y|^{n-2\alpha}}\left(\int_{|z|<|y|}\frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}}\right)^{\gamma_2}+\left(\int_{|z|\geq|y|}\frac{d\sigma(z)}{|z|^{n-2\alpha}}\right)^{\gamma_2}\right)$$

where λ is a constant to be determined later.

We claim that $\overline{u} \geq \mathbf{I}_{2\alpha}(\overline{v}^{q_1} d\sigma)$. For $x \neq 0$, we have

$$\begin{aligned}
\mathbf{I}_{2\alpha}(\overline{v}^{q_1} d\sigma) &\approx \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \overline{v}^{q_1} d\sigma + \int_{|y|\geq|x|} \frac{\overline{v}^{q_1} d\sigma}{|y|^{n-2\alpha}} \\
&\leq \lambda^{q_1} \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2\alpha)q_1}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{q_1\gamma_2} d\sigma(y) \\
&+ \lambda^{q_1} \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} d\sigma(y) \\
&+ \lambda^{q_1} \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \frac{1}{|y|^{(n-2\alpha)q_1}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{q_1\gamma_2} d\sigma(y) \\
&+ \lambda^{q_1} \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \left(\int_{|z|\geq|y|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} d\sigma(y) \\
&:= \lambda^{q_1}(I + II + III + IV).
\end{aligned}$$

Notice that

$$\begin{aligned}
I =& \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2\alpha)q_1}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1} |z|^{(n-2\alpha)(r_2-r_1)}} \right)^{q_1\gamma_2} d\sigma(y) \\
\leq& \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2\alpha)q_1(1+\gamma_2(r_2-r_1))}} \left(\int_{|z|<|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1}} \right)^{q_1\gamma_2} d\sigma(y) \\
\leq& \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}} \right)^{\gamma_1}.
\end{aligned}$$

For the sake of convenience, we'll split II in 2 parts:

$$\begin{aligned}
II =& \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} + \int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} d\sigma(y) \\
\leq& c \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{|y|\leq|z|<|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} d\sigma(y) \\
+& c, \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} d\sigma(y) \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} \\
=& c(II_a + II_b).
\end{aligned}$$

So

$$
\begin{aligned}
II_a &\leq \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \left(\int_{y \leq |z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1} |z|^{(n-2\alpha)(1-r_1)}} \right)^{q_1\gamma_2} d\sigma(y) \\
&\leq \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} \frac{1}{|y|^{(n-2\alpha)(1-r_1)q_1\gamma_2}} \left(\int_{y \leq |z| < |x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1}} \right)^{q_1\gamma_2} d\sigma(y) \\
&\leq \frac{1}{|x|^{n-2\alpha}} \left(\int_{|y|<|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}} \right)^{\gamma_1}
\end{aligned}
$$

and using Young's inequality with γ_1 and $\frac{1+q_1}{q_1+q_1q_2}$ we obtain we obtain

$$
\begin{aligned}
II_b &= \frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} d\sigma(y) \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} \\
&\leq c \left(\left(\frac{1}{|x|^{n-2\alpha}} \int_{|y|<|x|} d\sigma(y) \right)^{\gamma_1} + \left(\int_{|z|\geq|x|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{\gamma_1} \right).
\end{aligned}
$$

Next, we have

$$
\begin{aligned}
III &\leq c \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \frac{1}{|y|^{(n-2\alpha)q_1}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{q_1\gamma_2} d\sigma(y) \\
&+ c \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}} \frac{1}{|y|^{(n-2\alpha)q_1}} \left(\int_{|x|\leq|z|\leq|y|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{q_1\gamma_2} d\sigma(y) \\
&\leq c \frac{1}{|x|^{(n-2\alpha)q_1}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_2}} \right)^{q_1\gamma_2} \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\
&+ c \int_{|y|\geq|x|} \frac{1}{|y|^{n-2\alpha}|y|^{(n-2\alpha)q_1}} \left(\int_{|x|\leq|z|<|y|} \frac{|z|^{(n-2\alpha)(\frac{1}{\gamma_2})} d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} d\sigma(y) \\
&\leq c \frac{1}{|x|^{(n-2\alpha)q_1}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1}|z|^{(n-2\alpha)(r_2-r_1)}} \right)^{q_1\gamma_2} \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\
&+ c \left(\int_{|x|\leq|z|} \frac{d\sigma(z)}{|z|^{n-2\alpha}} \right)^{q_1\gamma_2} \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{(n-2\alpha)}} \\
&\leq c \frac{1}{|x|^{(n-2\alpha)q_1(1+\gamma_2(r_2-r_1))}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1}} \right)^{q_1\gamma_2} \int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \\
&+ c \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\gamma_1} \\
&\leq c \left(\frac{1}{|x|^{(n-2\alpha)}} \left(\int_{|z|<|x|} \frac{d\sigma(z)}{|z|^{(n-2\alpha)r_1}} \right)^{\gamma_1} + \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\gamma_1} \right) \\
&+ c \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\gamma_1}
\end{aligned}
$$

Where we applied Young's inequality in the last inequality. Finally, notice that

$$
IV \leq \left(\int_{|y|\geq|x|} \frac{d\sigma(y)}{|y|^{n-2\alpha}} \right)^{\gamma_1}.
$$

By choosing λ large enough we then guarantee that $\mathbf{I}_{2\alpha}(\overline{v}^{q_1} d\sigma) \leq \overline{u}$, and by symmetry it's also possible to conclude that $\mathbf{I}_{2\alpha}(\overline{u}^{q_2} d\sigma) \leq \overline{v}$.

To obtain a solution (u, v) we use the standard iteration argument of the sub-sup solution method of PDEs and the monotone convergence theorem. We reproduce the full argument for the convenience of the reader.

Let $u_0 = \underline{u}$ and $v_0 = \underline{v}$. Clearly, $u_0 \leq \overline{u}$ and $v_0 \leq \overline{v}$. Set $u_1 = \mathbf{I}_{2\alpha}(v_0^{q_1} \mathrm{d}\sigma)$ and $v_1 = \mathbf{I}_{2\alpha}(u_0^{q_2} \mathrm{d}\sigma)$. We have $u_1 \geq u_0$ and $v_1 \geq v_0$. We interate this process and obtain a sequence of pair of functions (u_j, v_j) such that

$$\begin{cases} u_j = \mathbf{I}_{2\alpha}(v_{j-1}^{q_1}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n, \\ v_j = \mathbf{I}_{2\alpha}(u_{j-1}^{q_1}\mathrm{d}\sigma) & \text{in} \quad \mathbb{R}^n, \end{cases} \tag{20}$$

By induction, the sequences $\{u_j\}$ and $\{v_j\}$ are nondecreasing, with $\underline{u} \leq u_j \leq \overline{u}$ and $\underline{v} \leq v_j \leq \overline{v}$ (for $j = 0, 1, \ldots$).

Using the Monotone Convergence Theorem and taking the limit as $j \to \infty$, we see that there exist nonnegative functions $u = \lim u_j$ and $v = \lim v_j$ such that (u, v) is a solution satisfying $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$. In particular, u and v satisfies (15).

□

3 Proof of theorem 1

Consider the following condition:

$$\begin{aligned} &\mathbf{I}_{2\alpha}\left((\mathbf{I}_{2\alpha}\sigma)^{q_1\gamma_2}\,\mathrm{d}\sigma\right) \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}\right), \\ &\mathbf{I}_{2\alpha}\left((\mathbf{I}_{2\alpha}\sigma)^{q_2\gamma_1}\,\mathrm{d}\sigma\right) \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right). \end{aligned} \tag{21}$$

According to (da Silva and do O, 2024, Thm. 1.3), these conditions are necessary and sufficient for the existence of solutions satisfying a Brezis-Kamin type estimates (9). Set

$$\begin{aligned} \mathbf{K_1}\sigma(x) &= \frac{1}{|x|^{n-2\alpha}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_1}}\right)^{\gamma_1}, \quad x \neq 0 \\ \mathbf{K_2}\sigma(x) &= \frac{1}{|x|^{n-2\alpha}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_2}}\right)^{\gamma_2}, \quad x \neq 0 \end{aligned} \tag{22}$$

Lemma 4. *Condition* (21) *is equivalent to*

$$\begin{aligned} \mathbf{K_1}\sigma(x) &\leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}\right) < \infty, \quad a.e. \\ \mathbf{K_2}\sigma(x) &\leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right) < \infty, \quad a.e. \end{aligned} \tag{23}$$

Proof. Suppose (21) holds. Then the there is a solution pair (u, v) satisfying (9), moreover by theorem 3 they satisfy:

$$\begin{aligned} \mathbf{K_1}\sigma(x) &\leq u \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}\right), \\ \mathbf{K_2}\sigma(x) &\leq v \leq c\left(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right), \end{aligned} \tag{24}$$

which implies (23).

Conversely, suppose (23) holds. Using theorem 3 again, we can guarantee the existence of a solution (u,v) satisfying:

$$\begin{aligned} u(x) &\leq c\,(\mathbf{K_1}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}) \leq c\,(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}) \\ v(y) &\leq c\,(\mathbf{K_2}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}) \leq c\,(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}) \end{aligned} \tag{25}$$

and by remark 2, we also have:

$$\begin{aligned} u(x) &= \mathbf{I}_{2\alpha}(v^{q_1}d\sigma) \geq \mathbf{I}_{2\alpha}\left((\mathbf{I}_{2\alpha}\sigma)^{q_1\gamma_2}\,\mathrm{d}\sigma\right) \\ v(x) &= \mathbf{I}_{2\alpha}(u^{q_2}d\sigma) \geq \mathbf{I}_{2\alpha}\left((\mathbf{I}_{2\alpha}\sigma)^{q_2\gamma_1}\,\mathrm{d}\sigma\right) \end{aligned} \tag{26}$$

which completes the proof. □

We conclude the proof of the theorem with the following lemma:

Lemma 5. *Condition* (23) *is equivalent to*

$$\int_{|y|\geq 1} \frac{d\sigma(y)}{|y|^{n-2\alpha}} < \infty \text{ and } \limsup_{|x|\to 0} \frac{\frac{1}{|x|^{(n-2\alpha)\frac{1}{\gamma_i}}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{r_i(n-2\alpha)}}}{\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2\alpha}}} < \infty, i=1,2. \tag{27}$$

Proof. Since $\lim_{|x|\to\infty}\mathbf{I}_{2\alpha}\sigma(x) = 0$, it's enough to see what happens when $\limsup_{|x|\to 0}\mathbf{I}_{2\alpha}\sigma(x) = \infty$, otherwise $\mathbf{I}_{2\alpha}\sigma$ is bounded and (21) holds trivially.

The proof is similar to the single equation case treated in (Cao and Verbitsky, 2016, Prop. 5.2). Suppose (27) holds, then for x sufficiently small, say $0 < |x| < \delta < 1$ we have:

$$\mathbf{K}_i\sigma(x) \leq c\,(\mathbf{I}_{2\alpha}\sigma(x))^{\gamma_i} \quad \text{for all } 0 < |x| < \delta.$$

If $0 < \delta < |x| < 1$ we have:

$$\mathbf{K}_i\sigma(x) \leq \frac{1}{\delta^{n-2\alpha}}\left(\int_{|y|<1}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}}\right)^{\gamma_i} < c\,\mathbf{I}_{2\alpha}\sigma(x)$$

Now, for $|x| > 1$:

$$\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}} \leq \int_{0\leq|y|\leq|1|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}} + \int_{1\leq|y|\leq|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}}$$

The first term is clearly bounded by $\mathbf{I}_{2\alpha}\sigma(x)$, we analyze the second term. Using Holder's inequality we obtain:

$$\int_{1\leq|y|\leq|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}} \leq \left(\int_{1\leq|y|\leq|x|}d\sigma(y)\right)^{1-\frac{1}{r_i}}\left(\int_{1\leq|y|\leq|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)}}\right)^{\frac{1}{r_i}}$$

Therefore,

$$\left(\int_{1\le|y|\le|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}}\right)^{\gamma_i}\le\left(\int_{|y|\le|x|}d\sigma(y)\right)^{\gamma_i(1-\frac{1}{r_i})}\left(\int_{1\le|y|\le|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)}}\right)^{\frac{\gamma_i}{r_i}}$$

and again we have:

$$\mathbf{K}_i\sigma(x)\le c\left(\mathbf{I}_{2\alpha}\sigma(x)\right)$$

Combining all cases together, we conclude that

$$\mathbf{K}_i\sigma(x)\le c\ \left(\mathbf{I}_{2\alpha}\sigma(x)+\left(\mathbf{I}_{2\alpha}\sigma(x)\right)^{\gamma_i}\right)<\infty$$

Now suppose (23) holds. Since we are assuming $\limsup_{|x|\to 0}\mathbf{I}_{2\alpha}\sigma(x)=\infty$, for x sufficiently small we have

$$\mathbf{I}_{2\alpha}\sigma(x)\le\left(\mathbf{I}_{2\alpha}\sigma(x)\right)^{\gamma_i},$$

hence

$$\mathbf{K}_i\sigma(x)\le c\left(\mathbf{I}_{2\alpha}\sigma(x)\right)^{\gamma_i},$$

or equivalently,

$$\frac{1}{|x|^{(n-2\alpha)\frac{1}{\gamma_i}}}\left(\int_{|y|<|x|}\frac{d\sigma(y)}{|y|^{(n-2\alpha)r_i}}\right)\le c\left(\frac{1}{|x|^{n-2\alpha}}\int_{|y|<|x|}d\sigma(y)+\int_{|y|\ge|x|}\frac{d\sigma(y)}{|y|^{n-2\alpha}}\right) \tag{28}$$

It suffices to estimate the first term. Let $d=f(|x|)$ be a function to be determined later, we have

$$\begin{aligned}\frac{1}{|x|^{n-2\alpha}}\int_{|y|<|x|}d\sigma(y)&=\frac{1}{|x|^{(n-2\alpha)(1-r_i)}|x|^{(n-2\alpha)r_i}}\int_{|y|<d}\frac{|y|^{(n-2\alpha)r_i}}{|y|^{(n-2\alpha)r_i}}d\sigma(y)\\&+\frac{1}{|x|^{n-2\alpha}}\int_{d<|y|<x}\frac{|y|^{n-2\alpha}}{|y|^{n-2\alpha}}d\sigma(y)\\&\le\frac{d^{(n-2\alpha)r_i}}{|x|^{(n-2\alpha)(1-r_i)}|x|^{(n-2\alpha)r_i}}\int_{|y|<d}\frac{1}{|y|^{(n-2\alpha)r_i}}d\sigma(y)\\&+\int_{d<|y|<x}\frac{d\sigma(y)}{|y|^{n-2\alpha}}\end{aligned} \tag{29}$$

Set $d=k|x|$, for k small enough we have

$$\begin{aligned}\frac{1}{|x|^{n-2\alpha}}\int_{|y|<|x|}d\sigma(y)\le\frac{k^{(n-2\alpha)r_i}}{|x|^{(n-2\alpha)(\frac{1}{\gamma_i})}}\int_{|y|<|x|}\frac{1}{|y|^{(n-2\alpha)r_i}}d\sigma(y)\\+\int_{|x|\le|y|}\frac{d\sigma(y)}{|y|^{n-2\alpha}}\end{aligned} \tag{30}$$

Together with (28), this implies that

$$\frac{\frac{1}{|x|^{(n-2\alpha)\frac{1}{\gamma_i}}}\int_{B(0,|x|)}\frac{d\sigma(y)}{|y|^{r_i(n-2\alpha)}}}{\int_{|y|\geq|x|}\frac{d\sigma(y)}{|y|^{n-2\alpha}}}<c,\quad i=1,2.$$

for some constant c and x sufficiently small, which completes the proof. □

Remark 3. *As explained in Cao and Verbitsky (2016) in the case $u = v$ and $q_1 = q_2 = q$, it's possible to find σ satisfying 14 but not (10). So we can have solutions that do not satisfy the Brezis-Kamin type estimates (9).*

4 Proof of Theorem 2

In this section we analyze the system:

$$\begin{cases} u = \mathbf{I}_{2\alpha}\left(v^{q_1}\mathrm{d}\sigma + d\mu_1\right), \\ v = \mathbf{I}_{2\alpha}\left(u^{q_2}\mathrm{d}\sigma + d\mu_2\right), \end{cases} \tag{31}$$

and give the proof of theorem 2.

The proof is based on the proof of (da Silva and do O, 2024, Thm 1) *mutatis mutandis* because of the condition $\mathbf{I}_{2\alpha}\mu_i \leq C\mathbf{I}_{2\alpha}\sigma$. Although not obvious, we'll see that this condition enables the existence of a supersolution.

A subsolution $(\underline{u}, \underline{v})$

Recall the following lemma from Cao and Verbitsky (2017) (simplified to our setting):
Lemma 6. *Let $\sigma \in M^+(\mathbb{R}^n)$. For every $r > 0$ and for all $x \in \mathbb{R}^n$, it holds*

$$\mathbf{I}_{2\alpha}\left((\mathbf{I}_{2\alpha}\sigma)^r\mathrm{d}\sigma\right)(x) \geq \kappa^r\left(\mathbf{I}_{2\alpha}\omega(x)\right)^{r+1}, \tag{32}$$

where κ depends only on n, p and α.

Consider the following pair of functions

$$(\underline{u}, \underline{v}) = \left(\lambda(\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}, \lambda(\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right),$$

where λ is a constant to be found. Then we have

$$\begin{aligned}
\mathbf{I}_{2\alpha}\left(\underline{v}^{q_1}\mathrm{d}\sigma + d\mu_1\right) &= \lambda^{q_1}\mathbf{I}_{2\alpha}((\mathbf{I}_{2\alpha}\sigma)^{q_1\gamma_2}\mathrm{d}\sigma + d\mu_1) \geq \lambda^{q_1}\left[\kappa^{q_1\gamma_2}(\mathbf{I}_{2\alpha}\sigma)^{q_1\gamma_2+1} + \mathbf{I}_{2\alpha}\mu_1\right] \\
&\geq \lambda^{q_1}\left[\kappa^{q_1\gamma_2}(\mathbf{I}_{2\alpha}\sigma)^{\gamma_1}\right], \\
\mathbf{I}_{2\alpha}\left(\underline{u}^{q_2}\mathrm{d}\sigma + d\mu_2\right) &= \lambda^{q_2}\mathbf{I}_{2\alpha}((\mathbf{I}_{2\alpha}\sigma)^{q_2\gamma_1}\mathrm{d}\sigma + d\mu_2) \geq \lambda^{q_2}\left[\kappa^{q_2\gamma_1}(\mathbf{I}_{2\alpha}\sigma)^{q_2\gamma_1+1} + \mathbf{I}_{2\alpha}\mu_2\right] \\
&\geq \lambda^{q_2}\left[\kappa^{q_2\gamma_1}(\mathbf{I}_{2\alpha}\sigma)^{\gamma_2}\right].
\end{aligned}$$

Therefore, by choosing λ sufficiently small we conclude that

$$\underline{u} \le \mathbf{I}_{2\alpha}(\underline{v}^{q_1}\,\mathrm{d}\sigma + d\mu_1),$$
$$\underline{v} \le \mathbf{I}_{2\alpha}(\underline{u}^{q_2}\,\mathrm{d}\sigma + d\mu_2).$$

A supersolution $(\overline{u}, \overline{v})$

Now we show that there exists $\lambda > 0$ sufficiently large such that

$$(\overline{u}, \overline{v}) = \big(\lambda\,(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_1})\,, \lambda\,(\mathbf{I}_{2\alpha}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2})\,\big)$$

is a supersolution to (31), such that $\overline{u} \ge \underline{u}$ and $\overline{v} \ge \underline{v}$. The following lemma from Cao and Verbitsky (2017) (simplified to our setting) will be needed

Lemma 7. *Let $\sigma \in M^+(\mathbb{R}^n)$ satisfying (12). Then, for every $s > 0$,*

$$\int_E (\mathbf{I}_{2\alpha}\sigma_E)^s \,\mathrm{d}\sigma \le c\,\sigma(E) \quad \textit{for all compact sets } E \subset \mathbb{R}^n, \tag{33}$$

where c is a positive constant. Moreover, if (33) holds for a given $s > 0$, then (12) holds; hence (33) holds for every $s > 0$.

We have

$$\begin{aligned}
\mathbf{I}_{2\alpha}(\overline{v}^{q_1}\,\mathrm{d}\sigma + d\mu_1)(x) &\le \lambda^{q_1}\left[c\int_0^\infty \frac{\int_{B(x,t)}(\mathbf{I}_{2\alpha}\sigma)^{q_1}\,\mathrm{d}\sigma + (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma}{t^{n-2\alpha}}\frac{\mathrm{d}t}{t} + \mathbf{I}_{2\alpha}\mu_1(x)\right] \\
&\le \lambda^{q_1}\left[c(\int_0^\infty \frac{\int_{B(x,t)}(\mathbf{I}_{2\alpha}\sigma)^{q_1}\,\mathrm{d}\sigma}{t^{n-2\alpha}}\frac{\mathrm{d}t}{t} + \int_0^\infty \frac{\int_{B(x,t)}(\mathbf{I}_{2\alpha}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma}{t^{n-2\alpha}}\frac{\mathrm{d}t}{t}) + C\,\mathbf{I}_{2\alpha}\sigma(x)\right] \\
&\le \lambda^{q_1}\left[c(I + II) + C\,\mathbf{I}_{2\alpha}\sigma(x)\right]
\end{aligned} \tag{34}$$

We estimate I and II separately. Notice that lemma 7 implies that

$$I \le c\,\mathbf{I}_{2\alpha}\sigma(x).$$

Before estimating II, consider the following estimate

$$\begin{aligned}
\int_{B(x,t)} (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2 q_1}\,\mathrm{d}\sigma &= \int_{B(x,t)} \Big[\int_0^\infty \Big(\frac{\sigma(B(y,r))}{r^{n-2\alpha}}\Big)\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y) \\
&\le c\int_{B(x,t)} \Big[\int_0^t \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y) \\
&\quad + c\int_{B(x,t)} \Big[\int_t^\infty \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big)\frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1}\mathrm{d}\sigma(y) =: c\,(II_1 + II_2)\,,
\end{aligned}$$

For $y \in B(x,t)$ and $r \leq t$, notice that $B(y,r) \subset B(x,2t)$, consequently

$$\begin{aligned} II_1 &= \int_{B(x,t)} \Big[\int_0^t \Big(\frac{\sigma(B(y,r))}{r^{n-\alpha p}}\Big) \frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \mathrm{d}\sigma(y) \\ &\leq \int_{B(x,2t)} \Big[\int_0^t \Big(\frac{\sigma(B(y,r)\cap B(x,2t))}{r^{n-2\alpha}}\Big) \frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \mathrm{d}\sigma(y) \\ &\leq \int_{B(x,2t)} \Big[\mathbf{I}_{2\alpha}\sigma_{B(x,2t)}\Big]^{\gamma_2 q_1} \mathrm{d}\sigma(y). \end{aligned}$$

Using (33) again, we deduce

$$II_1 \leq \int_{B(x,2t)} \big[\mathbf{I}_{2\alpha}\sigma_{B(x,2t)}\big]^{\gamma_2 q_1} \ \mathrm{d}\sigma(y) \leq c\,\sigma(B(x,2t)).$$

If $y \in B(x,t)$ and $r \geq t$ then $B(y,r) \subset B(x,2r)$, thus

$$\begin{aligned} II_2 &\leq \int_{B(x,t)} \Big[\int_t^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-2\alpha}}\Big) \frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \mathrm{d}\sigma(y) \\ &= \sigma(B(x,t))\Big[\int_t^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-2\alpha}}\Big) \frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \\ &\leq \sigma(B(x,t))\Big[\int_0^\infty \Big(\frac{\sigma(B(x,2r))}{r^{n-2\alpha}}\Big) \frac{\mathrm{d}r}{r}\Big]^{\gamma_2 q_1} \leq c\,\sigma(B(x,t))\,[\mathbf{I}_{2\alpha}\sigma(x)]^{\gamma_2 q_1}. \end{aligned}$$

Combining II_1 and II_2 we obtain

$$\int_{B(x,t)} (\mathbf{I}_{2\alpha}\sigma)^{\gamma_2 q_1}\, \mathrm{d}\sigma \leq c\, \left[\sigma(B(x,2t)) + (\mathbf{I}_{2\alpha}\sigma(x))^{\gamma_2 q_1}\, \sigma(B(x,t))\right],$$

and we finally conclude that

$$II \leq c(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma 1}),$$

so

$$\begin{aligned} \mathbf{I}_{2\alpha}(\overline{v}^{q_1}\mathrm{d}\sigma + d\mu_1)(x) &\leq \lambda^{q_1}(c(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma 1} + C\mathbf{I}_{2\alpha}\sigma(x)) \\ &\leq \lambda^{q_1}\, c\,(\mathbf{I}_{2\alpha}\sigma(x) + (\mathbf{I}_{2\alpha}\sigma)^{\gamma 1}) \end{aligned}$$

In conclusion, by choosing λ sufficiently large we have

$$\mathbf{I}_{2\alpha}(\overline{v}^{q_1}\mathrm{d}\sigma + d\mu_1)(x) \leq \overline{u}.$$

Similarly, the exact same reasoning for $\overline{u}, q_2, \gamma_2, \mu_2$ instead of $\overline{v}, q_1, \gamma_1, \mu_1$ should give that we can find λ such that

$$\mathbf{I}_{2\alpha}(\overline{u}^{q_2}\mathrm{d}\sigma + d\mu_2)(x) \leq \overline{v}.$$

We conclude that $(\overline{u}, \overline{v})$ is a supersolution. By arguing as in the proof of theorem 3, we arrive at a solution (u, v) satisfying

$$\underline{u} \leq u \leq \overline{u},$$
$$\underline{v} \leq v \leq \overline{v},$$

in particular (9) holds, which concludes the proof of theorem 2.

5 Some Open problems

We propose the following problems that could possibly generalize the ideas discussed here.

1. Is it still possible to use the method of sub-super solutions to solve the more general system below?

$$\begin{cases} (-\Delta)^{\alpha} u = \sigma\, u^{q_1} + \mu\, v^{r_1}, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ (-\Delta)^{\alpha} v = \sigma\, v^{q_2} + \mu\, v^{r_2}, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x| \to \infty} u(x) = 0, \quad \liminf\limits_{|x| \to \infty} v(x) = 0. \end{cases} \tag{35}$$

2. Let σ is radially symmetric and consider the system

$$\begin{cases} -\Delta_p u = \sigma\, v^{q_1}, & v > 0 \quad \text{in} \quad \mathbb{R}^n, \\ -\Delta_p v = \sigma\, u^{q_2}, & u > 0 \quad \text{in} \quad \mathbb{R}^n, \\ \liminf\limits_{|x| \to \infty} u(x) = 0, \quad \liminf\limits_{|x| \to \infty} v(x) = 0. \end{cases} \tag{36}$$

   What would be an equivalent condition to (10) in this case?

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