

# CONTINUOUS WEAK SOLUTIONS TO THE IDEAL HALL EQUATION

GENIVAL DA SILVA JR

ABSTRACT. Using the techniques introduced in [25], we construct continuous weak solutions to the ideal Hall equation with a given energy profile. This is a first step in the direction of proving the Onsager's conjecture for the more general Hall-MHD equations.

## 1. INTRODUCTION

During the past decade there have been a sizeable amount of research about constructing weak solutions to partial differential equations using convex integration [1, 2, 3, 5, 6, 7, 8, 9, 10, 25, 22, 23, 24, 26, 27, 28]. The idea of using convex integration to attack problems related to the existence of weak solutions started with C. de Lellis and L. Szekelyhidi Jr [22], when they showed the non uniqueness of weak solutions to the Euler equations:

$$(1) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla P = 0 \\ \nabla \cdot u = 0 \end{cases}$$

They were motivated by the similarity between this problem and the work of John Nash on short isometric imbeddings [30]. At the time of their work there was a famous open problem related to the Euler equations, namely, Onsager's conjecture on energy dissipation. In a few words, Onsager conjectured that a weak Holder continuous solution to the Euler equation would dissipate energy if the Holder exponent was less than  $\frac{1}{3}$  and would conserve it otherwise. The energy conservation part was proved many years ago by Constantin & Titi [15], using an estimate on commutators, the dissipation part on the other hand remained open until very recently when P. Isett [29] using convex integration techniques finally settled this question.

A generalization of the Euler equations is the Magnetohydrodynamic (MHD) system of equations given by:

$$(2) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + (B \cdot \nabla)B + \nabla p = \alpha \Delta u \\ \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u = \beta \Delta B \\ \nabla \cdot u = \nabla \cdot B = 0 \end{cases}$$

which models the behavior of electrically conducting fluids. Here  $u$  is the velocity field,  $B$  is the magnetic field and  $p$  a scalar pressure. This system combines Maxwell's equations of electromagnetism with the Navier-Stokes equations.

When one considers the Hall effect, mathematically given by  $\nabla \times ((\nabla \times B) \times B)$ , in a conducting fluid, we have what is called the *Hall-magnetohydrodynamics* (Hall-MHD) equations:

$$(3) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + (B \cdot \nabla)B + \nabla p = \alpha \Delta u \\ \partial_t B + (u \cdot \nabla)B - (B \cdot \nabla)u + \theta \nabla \times ((\nabla \times B) \times B) = \beta \Delta B \\ \nabla \cdot u = \nabla \cdot B = 0 \end{cases}$$

In this manuscript we will be interested in the case where  $u = \alpha = \beta = 0$  and  $\theta = 1$ . The resulting system is called *ideal Electron-magnetohydrodynamic* (EMHD) equation or simply the *ideal Hall equation*.

$$(4) \quad \begin{cases} \partial_t B + \nabla \times ((\nabla \times B) \times B) = 0 \\ \nabla \cdot B = 0 \end{cases}$$

This model and its generalization above have been intensely studied in the past decades [16, 18, 19] with many results on the existence and well-posedness of solutions. Very recently M. Dai [17] used convex integration with  $L_2$  scale to prove some interesting results concerning the EMHD system.

In these notes we'll use convex integration with Holder scale instead, closely following the same approach used in the same result for the Euler equations [25]. We won't provide a specific Holder exponent for the constructed solutions but we plan to do so and also discuss the same ideas for the whole Hall-MHD system in forthcoming works.

We will be interested in the so called magnetic energy, which is a function given by

$$(5) \quad E(t) = \|B\|_2^2 = \int_{\mathbb{T}^3} |B|^2(x, t) dx.$$

If we assume that  $B(x, t)$  is a classical solution, then taking the dot product with  $B$  on both sides of the first equation in 4 and integrating over  $\mathbb{T}^3$  gives:

$$(6) \quad \int_{\mathbb{T}^3} \partial_t B \cdot B dx + \int_{\mathbb{T}^3} \nabla \times ((\nabla \times B) \times B) \cdot B dx = 0$$

Notice that by the Divergence theorem we have:

$$(7) \quad \int_{\mathbb{T}^3} \nabla \times ((\nabla \times B) \times B) \cdot B dx = \int_{\mathbb{T}^3} ((\nabla \times B) \times B) \cdot \nabla \times B dx = 0$$

Therefore, we deduce from the above that

$$(8) \quad E'(t) = 2 \int_{\mathbb{T}^3} \partial_t B \cdot B dx = 0.$$

and the magnetic energy is conserved for classical solutions. This is natural, since the Hall equation resembles the Euler, and the latter also conserves energy in the case of classical solutions.

We are then led to the basic question

*If we assume lower regularity of the solutions is the energy still conserved?*

This is the content of the Onsager's conjecture, which was proved in the context of the Euler equation but to the best of our knowledge is not yet known for the Hall equation. The purpose of this manuscript is to address this question, or more generally the following conjecture:

**Conjecture 1.** (*Onsager's conjecture for the Hall equation*) *Let  $B(x, t) \in C^{0, \alpha}$  be a Holder continuous weak solution to the Hall equation. Then*

- a) *If  $\alpha > \frac{2}{3}$  the magnetic energy is conserved.*
- b) *If  $\alpha < \frac{2}{3}$  the magnetic energy is dissipated.*

The first part of the conjecture, when  $\alpha > \frac{2}{3}$ , has been known already [20]. We provide a simple proof which is a simple adaptation of the argument in [15].

A weak solutions to system 4 is a vector field  $B$  on  $[0, T] \times \mathbb{T}^3$  satisfying for any test functions  $\varphi \in C_c^\infty([0, T] \times \mathbb{T}^3)$ ,  $\phi \in C_c^\infty([0, T] \times \mathbb{T}^3, \mathbb{R}^3)$  the following system:

$$(9) \quad \int_0^T \int_{\mathbb{T}^3} B \cdot \phi_t + (B \otimes B) : \nabla(\nabla \times \phi) \, dxdt = 0$$

$$(10) \quad \int_0^T \int_{\mathbb{T}^3} B \cdot \nabla \varphi \, dxdt = 0$$

*Proof of Conjecture 1 a).* Let  $B$  be a weak-solution. Notice that whenever  $B$  is divergence free, we have  $\nabla \times ((\nabla \times B) \times B) = \nabla \times (\nabla \cdot (B \otimes B))$ . Mollify equations 4 to obtain:

$$(11) \quad \begin{cases} \partial_t B_l + \nabla \times (\nabla \cdot (B \otimes B)_l) = 0 \\ \nabla \cdot B_l = 0 \end{cases}$$

where  $B_l = B * \rho_l$  and  $\rho_l$  is a standard family of radially symmetric mollifiers of scale  $l$ . The above system is equivalent to

$$(12) \quad \begin{cases} \partial_t B_l + \nabla \times (\nabla \cdot (B_l \otimes B_l)) = \nabla \times (\nabla \cdot [(B_l \otimes B_l) - (B \otimes B)_l]) \\ \nabla \cdot B_l = 0 \end{cases}$$

Taking the scalar product with  $B_l$  of the first equation above we obtain

$$(13) \quad \partial_t B_l \cdot B_l + \nabla \times (\nabla \cdot (B_l \otimes B_l)) \cdot B_l = \nabla \times (\nabla \cdot [(B_l \otimes B_l) - (B \otimes B)_l]) \cdot B_l$$

Integrating the above leads to

$$(14) \quad \frac{d}{dt} \int_{\mathbb{T}^3} |B_l|^2 dx = \int_{\mathbb{T}^3} [(B_l \otimes B_l) - (B \otimes B)_l] : \nabla(\nabla \times B_l) dx$$

Standard Mollifiers properties give us the bound:

$$(15) \quad |\nabla(\nabla \times B_l)| \leq C \|B\|_\alpha l^{\alpha-2}$$

and the main commutator estimate of [15] gives:

$$(16) \quad |(B_l \otimes B_l) - (B \otimes B)_l| \leq C \|B\|_\alpha^2 l^{2\alpha}$$

We conclude that

$$(17) \quad \left| \frac{d}{dt} \int_{\mathbb{T}^3} |B_l|^2 dx \right| \leq C \|B\|_\alpha^3 l^{3\alpha-2}$$

and statement a) is proved.  $\square$

Similarly to the case of the Euler equation, the second part of conjecture 1 is non trivial and we hope to provide a proof in the future. In this manuscript we'll restrict our efforts to prove the following result which is a step in the direction of part b) of conjecture 1.

**Theorem 1.1.** *Given a function  $E : [0, 1] \rightarrow \mathbb{R}$  smooth and positive. There is a continuous vector field  $B : \mathbb{T}^3 \times [0, 1] \rightarrow \mathbb{R}^3$  which solve the ideal EMHD equations 4 in the sense of distributions and such that*

$$(18) \quad E(t) = \int_{\mathbb{T}^3} |B|^2(x, t) dx.$$

We assume the existence of a vector potential  $A$  such that  $B = \nabla \times A$ . Then the equations 4 implies that  $A$  satisfies:

$$(19) \quad \begin{cases} \partial_t A + (\nabla \times B) \times B = 0 \\ \nabla \cdot B = 0 \end{cases}$$

where  $A$  can be recovered from  $B$  by the Biot-Savart law  $A = \nabla \times (-\Delta^{-1})B$ .

The main tool to prove theorem 1.1 is to use convex integration in the relaxed system:

$$(20) \quad \begin{cases} \partial_t A_q + \nabla \cdot (B_q \otimes B_q) - \nabla p_q = \nabla \cdot \mathring{R}_q \\ \nabla \times A_q = B \\ \nabla \cdot A_q = \nabla \cdot B = 0 \end{cases}$$

where  $p_q = \frac{|B_q|^2}{2}$  and  $\mathring{R}_q$  is a symmetric traceless matrix tensor field.

The main theorem is a consequence of the following proposition.

**Proposition 1.2.** *(Main proposition) Let  $E(t)$  be as in Theorem 1.1. Then there are positive constants  $\eta$  and  $M$  with the following property.*

*Let  $\delta \leq 1$  be any positive number and  $(A, B, R)$  be a solution system 20 such that for every  $t \in [0, 1]$ :*

$$(21) \quad \frac{3\delta}{4}E(t) \leq E(t) - \int_{\mathbb{T}^3} |B|^2(x, t)dx \leq \frac{5\delta}{4}E(t)$$

$$(22) \quad \|\mathring{R}_q\|_0 \leq \eta\delta$$

*Then there is a triple  $(A_1, B_1, \mathring{R}_1)$  which solves system 20 as well and satisfies:*

$$(23) \quad \frac{3\delta}{8}E(t) \leq E(t) - \int_{\mathbb{T}^3} |B_1|^2(x, t)dx \leq \frac{5\delta}{8}E(t)$$

$$(24) \quad \|\mathring{R}_1\|_0 \leq \eta\frac{\delta}{2}$$

$$(25) \quad \|B_1 - B\|_0 \leq M\sqrt{\delta}$$

*Proof of theorem 1.1.* The proof is almost identical to the one found in [25, Proposition 2.2], we include here for the convenience of the reader. Set  $(A_0, B_0, \mathring{R}_0) = (0, 0, 0)$ , take  $\delta = 1$  and apply proposition 1.2 iteratively. We get sequences  $(A_n, B_n, \mathring{R}_n)$  satisfying

$$(26) \quad \frac{3}{4} \frac{E(t)}{2^n} \leq E(t) - \int_{\mathbb{T}^3} |B_n|^2(x, t)dx \leq \frac{5}{4} \frac{E(t)}{2^n}$$

$$(27) \quad \|\mathring{R}_n\|_0 \leq \frac{\eta}{2^n}$$

$$(28) \quad \|B_{n+1} - B_n\|_0 \leq M\sqrt{\frac{1}{2^n}}$$

$$(29)$$

We have that  $B_n$  is a Cauchy sequences in  $C(\mathbb{T}^3 \times [0, 1])$  and therefore converge uniformly to a continuous  $B$ , moreover, if that's the case then  $A_n$  converges to a continuous  $A$  as well, such that  $\nabla \cdot A = 0$  and  $\nabla \times A = B$ . Also notice that  $\mathring{R}_n$  converges uniformly to 0. We conclude that  $B$  defines a continuous weak solution to the EMHD equations 4 satisfying

$$E(t) = \int_{\mathbb{T}^3} |B|^2(x, t)dx$$

□

## 2. CONSTRUCTION OF THE PERTURBATIONS

In order to construct the perturbations, we will need the following lemma whose proof can be found in [25, Lemma 3.2].

**Lemma 2.1.** *For every  $N \in \mathbb{N}$  we can choose  $r_0 > 0$  and  $\lambda_0 > 1$  with the following property. There exist pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^3 : |k| = \lambda_0\} \quad j \in \{1, \dots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(Id)) \quad j \in \{1, \dots, N\}, k \in \Lambda_j$$

such that

- a) If  $k \in \Lambda_j$  then  $-k \in \Lambda_j$  and  $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$ ;
- b) For each  $R \in C^\infty(B_{r_0}(Id))$  we have the identity

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} (\gamma_k^{(j)}(R))^2 \left( Id - \frac{k}{|k|} \otimes \frac{k}{|k|} \right)$$

There is a strong similarity between the relaxed system for the EMHD equations 20, and the one for the Euler equations  $\partial_t u + (u \cdot \nabla)u - \nabla P = 0$  of an ideal incompressible fluid, which in latter case is given by

$$(30) \quad \begin{cases} \partial_t u_q + \nabla \cdot (u_q \otimes u_q) + \nabla P_q = \nabla \cdot S_q \\ \nabla \cdot u_q = 0 \end{cases}$$

Based on this similarity we construct perturbations for our convex integration scheme following the ideas of [25]. Namely, using Beltrami flows. The upshot is that if  $\sum_{|k|=\lambda_0} a_k(x) B_k e^{i\lambda k \cdot x}$  is a Beltrami flow for some  $\lambda_0$  then

$$(31) \quad \frac{1}{\lambda} \sum_{|k|=\lambda_0} a_k B_k e^{i\lambda k \cdot x} - \frac{1}{\lambda^2} \sum_{|k|=\lambda_0} i \nabla a_k(x) \times \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x} = \frac{1}{\lambda^2} \nabla \times \left( \sum_{|k|=\lambda_0} -i a_k(x) \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x} \right)$$

We fix constants  $1 \leq \mu \leq \lambda$  to be defined later. Throughout these notes our amplitude function  $a_k$  in the expression for the perturbations will be very similar to the one used in [25], namely  $a_k(x, t, \lambda t)$  where  $a_k$  is defined in our case by

$$(32) \quad \sum_{|k|=\lambda_0} a_k(y, s, \tau) B_k e^{i\lambda k \cdot x} = \rho(s) \sum_{j=1}^8 \sum_{k \in \Lambda_j} \gamma_k^{(j)} \left( \frac{R(y, s)}{\rho(s)} \right) \phi_k^{(j)}(B(y, s), \tau) B_k e^{i\lambda k \cdot x}$$

$$(33) \quad R(y, s) = \rho(s) Id - \mathring{R}(y, s)$$

$$(34) \quad \rho(s) = \frac{1}{(2\pi)^3} \left( E(s) \left( 1 - \frac{\delta}{2} \right) - \int_{\mathbb{T}^3} |B|^2(x, s) dx \right)$$

and  $\lambda_0, \gamma_k^{(j)}$  can be obtained from lemma 2.1 by setting  $N = 8$ , and  $\phi_k^{(j)}(B, \tau)$  is a type of partition of unity satisfying  $\sup_{B, \tau} |D_B^m \phi_k^{(j)}(B, \tau)| \leq C(m) \mu^m$ , for the definition of  $\phi_k^{(j)}(B, \tau)$  see [25].

Let  $\mathbb{P}$  denote the mean zero Leray Projection on Divergence free vector fields. Motivated by equation 31 we define  $A_1$  to be:

$$(35) \quad A_1 = A + \mathbb{P}(v) = A + v + v_c$$

$$(36) \quad v = \frac{1}{\lambda^2} \left( \sum_{|k|=\lambda_0} -i a_k(x, t, \lambda t) \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x} \right)$$

$$(37) \quad v_c = -\mathcal{Q}(v)$$

$$(38)$$

where  $\mathcal{Q}$  is the operator defined by  $Id - \mathbb{P}$  (see [25] for details). Similarly, we have the definition of  $B_1$ :

$$(39) \quad B_1 = \nabla \times A_1$$

$$(40) \quad = \nabla \times (A + v - \nabla \phi - \int_{\mathbb{T}^3} v)$$

$$(41) \quad = B + \frac{1}{\lambda} \sum_{|k|=\lambda_0} a_k B_k e^{i\lambda k \cdot x} - \frac{1}{\lambda^2} \sum_{|k|=\lambda_0} i \nabla a_k(x) \times \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x}$$

In order to simplify notation, we set

$$w_o := \frac{1}{\lambda} \sum_{|k|=\lambda_0} a_k B_k e^{i\lambda k \cdot x}$$

$$w_c := -\frac{1}{\lambda^2} \sum_{|k|=\lambda_0} i \nabla a_k(x) \times \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x}$$

$$B_1 = B + w_o + w_c$$

Notice that  $\nabla \cdot A_1 = \nabla \cdot B_1 = 0$  and  $B_1 = \nabla \times A_1$  by construction. Also, in order for  $a_k$  to make sense we need

$$\left\| \frac{R(y, s)}{\rho(s)} - Id \right\| \leq r_0$$

where  $r_0$  is given by lemma 2.1. Motivated by this we set, set  $m := \min E(t)$  then  $\rho(t) \geq c\delta m$  for some numerical constant  $c$ . So if we define

$$\eta := \frac{cmr_0}{2}$$

the amplitude functions  $a_k$  are well-defined. Similarly, recall that we choose  $\lambda \geq 1$  and by definition we have  $\rho(t) \leq \delta E(t)$ , so we can choose a numerical constant  $M > 1$  depending only on  $E(t)$  such that  $\|w_o\| \leq \frac{\sqrt{M\delta}}{2}$ .

### 3. ESTIMATES

**Lemma 3.1.** *(Estimate on the energy) The energy satisfies the following estimate*

$$(42) \quad \left| E(t) \left(1 - \frac{\delta}{2}\right) - \int_{\mathbb{T}^3} |B_1|^2(x, t) dx \right| \leq C \frac{\mu}{\lambda^{1-\alpha}}$$

*Proof.* The proof follows from the identity

$$(43) \quad w_o \otimes w_o = R(x, t) + \sum_{0 \leq |k| \leq 2\lambda_0} U_k(x, t, \lambda t) e^{i\lambda k \cdot x}$$

where  $U_k(y, s, \tau)$  satisfies

$$\|U_k(\cdot, s, \tau)\|_r \leq C\mu^r$$

Taking the trace of equation 43 we get

$$|w_o|^2 = 3\rho(x, t) + \sum_{0 \leq |k| \leq 2\lambda_0} c_k(x, t, \lambda t) e^{i\lambda k \cdot x}$$

for some  $c_k(y, s, \tau)$  such that

$$\|c_k(\cdot, s, \tau)\|_r \leq C\mu^r$$

Thus, by proposition [??] we have

$$\left| \int_{\mathbb{T}^3} |w_o|^2 - \text{tr}(R) \, dx \right| \leq C \frac{\mu}{\lambda}$$

Additionally, we have

$$\left| \int_{\mathbb{T}^3} |B_1|^2 - |B|^2 - |w_o|^2 \, dx \right| \leq C \frac{\mu}{\lambda^{1-\alpha}}$$

The proof follows by combining these two inequalities.  $\square$

#### 4. ESTIMATE OF THE STRESS TENSOR

After the following lemma, we'll have all the tools needed to estimate  $\mathring{R}_1$ .

**Lemma 4.1.** [25, Collorary 5.3] *Let  $k \in \mathbb{Z}^3 \setminus \{0\}$  be fixed. For a smooth vector field  $a \in C^\infty(\mathbb{T}^3, \mathbb{R}^3)$  let  $F(x) := a(x)e^{i\lambda \cdot k}$ . Then we have:*

$$\|\mathcal{R}(F)\|_\alpha \leq \frac{C}{\lambda^{1-\alpha}} \|a\|_0 + \frac{C}{\lambda^{m-\alpha}} [a]_m + \frac{C}{\lambda^m} [a]_{m+\alpha}$$

Recall that  $\mathring{R}_1$  has to satisfy the equation:

$$(44) \quad \partial_t A_1 + \nabla \cdot (B_1 \otimes B_1) - \nabla \frac{|B_1|^2}{2} = \nabla \cdot \mathring{R}_1$$

and notice that

$$\begin{aligned} \partial_t A_1 + \nabla \cdot (B_1 \otimes B_1) - \nabla \frac{|B_1|^2}{2} &= \partial_t v + (B \cdot \nabla) w_o \\ &+ \nabla \cdot (w_o \otimes w_o - \frac{1}{2} |w_o|^2 Id + \mathring{R}) - \frac{1}{2} \nabla (2B \cdot w_o + 2w_o \cdot w_c + |w_c|^2) \\ &+ \partial_t v_c + \nabla \cdot (B_1 \otimes w_c + w_c \otimes B_1 - w_c \otimes w_c + B \otimes w_o) \\ &=: R_{transport} + R_{oscillation} + R_{error} + \nabla \mathring{p}_{q+1} \end{aligned}$$

**Lemma 4.2.** (*transport error*)

$$(45) \quad \|\mathcal{R}(\partial_t v + (B \cdot \nabla) w_o)\|_\alpha \leq C \left( \frac{\mu^2}{\lambda^{3-\alpha}} + \frac{\mu}{\lambda^{1-\alpha}} + \frac{\mu^2}{\lambda^{2-\alpha}} \right)$$

*Proof.* Recall that  $ik \times B_k = \lambda_0 B_k$ . We have:

$$\begin{aligned} \mathcal{R}(\partial_t v + (B \cdot \nabla) w_o) &= \frac{1}{\lambda^2} \mathcal{R} \left( - \sum_{|k|=\lambda_0} \partial_s a_k(x, t, \lambda t) \frac{B_k}{|k|^2} e^{i\lambda k \cdot x} \right) \\ &+ \mathcal{R} \left( \sum_{|k|=\lambda_0} (i(k \cdot B) a_k)(x, t, \lambda t) B_k e^{i\lambda k \cdot x} \right) \\ &+ \frac{1}{\lambda} \mathcal{R} \left( \sum_{|k|=\lambda_0} \left( \frac{\lambda_0}{|k|^2} \partial_\tau a_k + B \cdot \nabla_y a_k \right) (x, t, \lambda t) B_k e^{i\lambda k \cdot x} \right) \end{aligned}$$

Using Lemma 4.1 with  $m = 1$  in the first expression we have the bound

$$\frac{\mu}{\lambda^{3-\alpha}} + \frac{\mu^2}{\lambda^{3-\alpha}} + \frac{\mu^{3+\alpha}}{\lambda^3}$$

the same lemma with  $m = 1$ , applied to the second expression gives the bound

$$\frac{C}{\lambda^{1-\alpha}} + \frac{\mu}{\lambda^{1-\alpha}} + \frac{\mu^{\alpha+1}}{\lambda}$$

and finally  $m = 1$  in the last expression gives:

$$\frac{\mu}{\lambda^{2-\alpha}} + \frac{\mu^2}{\lambda^{2-\alpha}} + \frac{\mu^{1+\alpha}}{\lambda^2}$$

Since we choose  $\lambda \geq \mu \geq 1$ , the result follows.  $\square$

The following estimate is identical to [25, lemma 7.3], so we omit the proof.

**Lemma 4.3.** (*Oscillation error*)

$$(46) \quad \|\mathcal{R}(\nabla \cdot (w_o \otimes w_o - \frac{1}{2}|w_o|^2 Id + \dot{R}))\|_\alpha \leq C \left( \frac{\mu^2}{\lambda^{1-\alpha}} \right)$$

**Lemma 4.4.** (*Estimate on  $w_c$* )

$$(47) \quad \|w_c\|_\alpha \leq C \frac{\mu}{\lambda^{2-\alpha}}$$

*Proof.* Follows directly from the expression

$$w_c = -\frac{1}{\lambda^2} \sum_{|k|=\lambda_0} i \nabla a_k(x) \times \frac{k \times B_k}{|k|^2} e^{i\lambda k \cdot x}$$

$\square$

**Lemma 4.5.** (*Estimate of  $v_c$* )

$$(48) \quad \|\mathcal{R}(\partial_t v_c)\| \leq C \left( \frac{\mu^2}{\lambda^{(3-\alpha)}} + \frac{\mu}{\lambda^{(2-\alpha)}} \right)$$

*Proof.* First notice that  $\mathcal{R}(\partial_t v_c)$  is  $\mathcal{R}(\mathcal{Q}(-\partial_t v))$  and

$$\partial_t v = \frac{\lambda_0}{\lambda^2} \left( - \sum_{|k|=\lambda_0} \partial_s a_k(x, t, \lambda t) \frac{B_k}{|k|^2} e^{i\lambda k \cdot x} \right) + \frac{\lambda_0}{\lambda} \left( - \sum_{|k|=\lambda_0} \partial_\tau a_k(x, t, \lambda t) \frac{B_k}{|k|^2} e^{i\lambda k \cdot x} \right)$$

The estimate follows from 4.1 and the fact that  $\mathcal{R}\mathcal{Q}$  is bounded with respect to  $\mathcal{C}^{m,\alpha}$  [25, Proposition 5.1].  $\square$



**Lemma 4.6.** (*Estimate on the error*)

$$(49) \quad \|\mathcal{R}(\nabla \cdot (B_1 \otimes w_c + w_c \otimes B_1 - w_c \otimes w_c))\|_\alpha \leq C \frac{\mu}{\lambda^{(3-2\alpha)}}$$

*Proof.* We have:

$$(50)$$

$$(51) \quad \begin{aligned} \|B_1 \otimes w_c + w_c \otimes B_1 - w_c \otimes w_c\|_\alpha &\leq C(\|B_1\|_0 \|w_c\|_\alpha + \|B_1\|_\alpha \|w_c\|_0 + \|w_{q+1}\|_0 \|w_{q+1}\|_\alpha) \\ &\leq C \frac{\mu}{\lambda^{2-\alpha}} (\|B_1\|_\alpha + \|w_c\|_\alpha) \end{aligned}$$

$$(52) \quad \leq C \frac{\mu}{\lambda^{2-\alpha}} (\|B\|_\alpha + \|w_o\|_\alpha + \|w_c\|_\alpha)$$

$$(53) \quad \leq C \frac{\mu}{\lambda^{2-\alpha}} \left( C + C \frac{\mu}{\lambda^{2-\alpha}} + \frac{C}{\lambda^{1-\alpha}} \right)$$

Recall that  $\lambda \geq \mu \geq 1$ , hence  $\frac{\mu}{\lambda^{2-\alpha}} \leq \frac{1}{\lambda^{1-\alpha}}$ .  $\square$

**Lemma 4.7.** (*Estimate on last part of the error*)

$$(54) \quad \|\mathcal{R}(\nabla \cdot (B \otimes w_o))\|_\alpha \leq C \frac{\mu^2}{\lambda^{2-\alpha}}$$

*Proof.* We have:

$$(55) \quad \nabla \cdot (B \otimes w_o) = (w_o \cdot \nabla)B + (\nabla \cdot w_o)B$$

$$(56) \quad = \frac{1}{\lambda} \sum_{|k|=\lambda_0} [a_k (B_k \cdot \nabla)B + B(B_k \cdot \nabla a_k)] e^{i\lambda k \cdot x}$$

the result follows by 4.1 with  $m = 1$ .  $\square$

## 5. PROOF OF THE MAIN PROPOSITION

Suppose we choose  $\mu = \lambda^b$  in such a way that  $\frac{\lambda}{\mu} \in \mathbb{N}$  (due to the definition of  $a_k$ ). Collecting all the estimates we've obtained so far, we obtain:

$$(57) \quad \|w_o + w_c\|_0 \leq C \left( \frac{M\sqrt{\delta}}{2\lambda} + \frac{\mu}{\lambda^{2-\alpha}} \right)$$

$$(58) \quad \|\mathring{R}_1\| \leq C \left( \lambda^{b-(2-\alpha)} + \lambda^{2b-(2-\alpha)} + \lambda^{b-(3-2\alpha)} + \lambda^{2b-(1-\alpha)} + \lambda^{2b-(3-\alpha)} + \lambda^{b-(1-\alpha)} \right)$$

Hence, any choice of  $\alpha, b$  satisfying

$$(59) \quad 0 < \alpha \leq 1$$

$$(60) \quad b < \frac{1-\alpha}{2}$$

will force  $\|\mathring{R}_1\|_0 \leq \eta \frac{\delta}{2}$  for sufficiently large  $\lambda$ , so that 24 is satisfied, and also  $\|w_o + w_c\|_0 \leq M\sqrt{\delta}$ , hence 25 holds.

Finally, according to 42, 23 holds provided that we can find  $\alpha, b$  such that  $C\lambda^{\alpha+b-1} \leq \frac{\sqrt{\delta}}{2}$ . That is indeed the case if we take  $\lambda$  large enough and  $\alpha, b$  as described by the inequalities above. This completes the proof of the main proposition.

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DEPARTMENT OF MATHEMATICS & COMPUTER SCIENCE  
EASTERN ILLINOIS UNIVERSITY  
CHARLESTON, IL

*E-mail:* [gfernandesdasilvaj@eiu.edu](mailto:gfernandesdasilvaj@eiu.edu)