On the limiting behavior of variations of Hodge structures

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Introduction

In this presentation I will address three results concerning the limiting behavior of variations of Hodge structures. The first result discuss extensions classes representing LMHS, I will compute them for a certain class of toric families. The next result is concerned with the so called Apéry constants, I will provide a method of computing such constants by using higher normal functions coming from geometry. Finally, in the last result I will analyze a family of surfaces with geometric monodromy group G_2.

Mirror symmetry and CY-variations of Hodge structures

Until recently, toric mirror symmetry only identified *complex* variations of Hodge structure arising from the A-model and B-model, because the Dubrovin connection on quantum cohomology merely provides a $\mathbb{C}$-local system on the A-model side. Iritani's mirror theorem says that the integral structure on this local system provided by the $\hat{\Gamma}$-class (in the sense I will describe soon) completes the A-model $\mathbb{C}$-VHS to a $\mathbb{Z}$-VHS matching the one arising from H^3 of fibers on the B-model side. The upshot is that to compute Ω_{lim} (at 0) for a 1-parameter family of toric complete intersection Calabi-Yau 3-folds $X_t \subset \mathbb{P}_\Delta$ over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, we may use what boils down to characteristic class data from the mirror $X_t^\circ \subset \mathbb{P}_{\Delta^\circ}$.

Mirror symmetry and CY-variations of Hodge structures

In each case:

- $V := H^{even}(X^\circ, \mathbb{C}) = \oplus_{j=0}^{3} H^{j,j}(X^\circ)$ is a vector space of rank 4.
- $\mathbb{P} := \mathbb{P}_{\Delta^\circ} = \mathbb{WP}(\delta_0, \ldots, \delta_{3+r})$ is a weighted projective space (with $\delta_0 = \delta_1 = 1$).
- $X^\circ \subset \mathbb{P}$ is smooth of multidegree $(d_k)_{k=1}^{r}$ with $\sum d_k =$ $= \sum \delta_i =: m$.
- H denotes the intersection with X° of the vanishing locus of the coordinate X_0.
- $\tau[H] \in H^{1,1}(X^\circ)$ denotes the Kähler class
- $q = e^{2\pi i \tau}$ for the Kähler parameter

We shall give a general recipe (following Doran-Kerr) for constructing a polarized $\mathbb{Z}$-VHS, over $\Delta^* : \; 0 < |q| < \epsilon$, on $\mathcal{V} := V \otimes \mathcal{O}_{\Delta^*}$.

Mirror symmetry and CY-variations of Hodge structures

The easy parts are the Hodge filtration and polarization. Indeed, we simply put:

- $F^p := \oplus_{j \leq 3-p} H^{j,j} \subset V$
- $\mathcal{F}_e^p := F^p \otimes \mathcal{O}_\Delta \subset V \otimes \mathcal{O}_\Delta =: \mathcal{V}_e$.
- Q on $\mathcal{V}_e$ is induced from the form on V given by the direct sum of pairings

$$Q_j : H^{j,j} \times H^{3-j,3-j} \to \mathbb{C}$$

defined by $Q_j(\alpha, \beta) := (-1)^j \int_{X^\circ} \alpha \cup \beta$.

A Hodge basis $e = \{e_i\}_{i=0}^3$ of H^{even}, with $e_i \in H^{3-i,3-i}(X^\circ)$ and $[Q]_e$ of the form (1), is given by $e_3 = [X^\circ]$, $e_2 = [H]$, $e_1 = -[L]$, and $e_0 = [p]$. Here L is a copy of $\mathbb{P}^1$ (parametrized by $[X_0 : X_1]$) in X° with $L \cdot H = p$, and $[H] \cdot [H] = m[L]$. The $\{e_i\}$ give a Hodge basis for $\mathcal{V}_e$.

$$Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad (1)$$

Mirror symmetry and CY-variations of Hodge structures

For the local system, we consider the generating series

$$\Phi_h(q) := \frac{1}{(2\pi i)^3} \sum_{d \geq 1} N_d q^d$$

of the genus-zero Gromov-Witten invariants of X°, and define the small quantum product on V by

- $e_2 * e_2 := -(m + \Phi_h'''(q))e_1$
- $e_i * e_j := e_i \cup e_j, \quad (i,j) \neq (2,2)$

This gives rise to the Dubrovin connection

$$\nabla := \mathrm{id}_V \otimes d + (e_2 *) \otimes d\tau,$$

which we view as a map from $\mathcal{V} \cong V \otimes \mathcal{O}_{\Delta^*} \to V \otimes \Omega^1_{\Delta^*} \cong \mathcal{V} \otimes \Omega^1_{\Delta^*}$, and the $\mathbb{C}$-local system $\mathbb{V}_{\mathbb{C}} := \ker(\nabla) \subset \mathcal{V}$.

Mirror symmetry and CY-variations of Hodge structures

Now define a map $\tilde{\sigma} : V \to V \otimes \mathcal{O}(\Delta)$ by

$$\tilde{\sigma}(e_0) := e_0,\ \tilde{\sigma}(e_1) := e_1,\ \tilde{\sigma}(e_2) := e_2 + \Phi_h'' e_1 + \Phi_h' e_0,$$

$$\tilde{\sigma}(e_3) := e_3 + \Phi_h' e_1 + 2\Phi_h e_0.$$

For any $\alpha \in V$, one easily checks that

$$\sigma(\alpha) := \tilde{\sigma}\left(e^{-\tau[H]} \cup \alpha\right) := \sum_{k \geq 0} \frac{(-1)^k \tau^k}{k!} \tilde{\sigma}\left([H]^k \cup \alpha\right)$$

satisfies $\nabla\sigma(\alpha) = 0$, hence yields an isomorphism $\sigma : V \xrightarrow{\cong} \Gamma(\mathfrak{H}, \rho^* \mathbb{V}_{\mathbb{C}})$ (where $\rho : \mathfrak{H} \to \Delta^*$ sends $\tau \mapsto q$).

Mirror symmetry and CY-variations of Hodge structures

Writing

$$\hat{\Gamma}(X^\circ) := \exp\left(-\frac{1}{24}ch_2(X^\circ) - \frac{2\zeta(3)}{(2\pi i)^3}ch_3(X^\circ)\right) \in V,$$

The image of

$$\begin{array}{cccc} \gamma: & K_0^{num}(X^\circ) & \longrightarrow & \Gamma(\mathfrak{H}, \rho^*\mathbb{V}_{\mathbb{C}}) \\ & \xi & \mapsto & \sigma(\hat{\Gamma}(X^\circ) \cup ch(\xi)) \end{array}$$

defines Iritani's $\mathbb{Z}$-local system $\mathbb{V}$ underlying $\mathbb{V}_{\mathbb{C}}$.
The filtration $W_\bullet := W(N)_\bullet$ associated to its monodromy $T(\gamma(\xi)) = \gamma(\mathcal{O}(-H) \otimes \xi)$ satisfies $W_k\mathcal{V}_e = \left(\oplus_{j \geq 3-k/2} H^{j,j}\right) \otimes \mathcal{O}_\Delta$.

Mirror symmetry and CY-variations of Hodge structures

In order to compute the limiting period matrix(following GGK) of this $\mathbb{Z}$-VHS over Δ^*, we shall require a (multivalued) basis $\{\gamma_i\}_{i=0}^3$ of $\mathbb{V}$ satisfying:

- $\gamma_i \in W_{2i} \cap \mathbb{V}$
- $\gamma_i \equiv e_i \mod W_{2i-2}$
- $[Q]_\gamma = [Q]_e$

Set $\tilde{\mathbb{V}} = j_*(e^{\frac{log(s)}{2\pi i}N}\mathbb{V})$, where $\Delta^* \overset{j}{\subset} \Delta$. The corresponding $\mathbb{Q}$-basis of $\tilde{\mathbb{V}}|_{q=0} =: V_{lim}$ is given by $\gamma_i^{lim} := \tilde{\gamma}_i(0)$ where $\tilde{\gamma}_i := e^{-\tau N}\gamma_i \in \Gamma(\Delta, \tilde{\mathbb{V}})$. Of course, the e_i are another basis of $V_{lim,\mathbb{C}}$, and

$$\Omega_{lim} = {}_{\gamma^{lim}}[\mathrm{id}]_e$$

Since $N_{lim} = -(2\pi i)Res_{q=0}(\nabla) = -(e_2*)|_{q=0} = -(e_2\cup)|_{q=0}$, we have

$$[N_{lim}]_e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Mirror symmetry and CY-variations of Hodge structures

A basis of the form we require is obtained by considering the Mukai pairing

$$\langle \xi, \xi' \rangle := \int_{X^\circ} ch(\xi^\vee \otimes \xi') \cup Td(X^\circ)$$

on $K_0^{num}(X^\circ)$. Since $\langle \xi, \xi' \rangle = Q(\gamma(\xi), \gamma(\xi'))$, any Mukai-symplectic basis of $K_0^{num}(X^\circ)$ of the form

$$\begin{aligned} \xi_1 &= \mathcal{O} + A\mathcal{O}_H + B\mathcal{O}_L + C\mathcal{O}_p \\ \xi_2 &= \mathcal{O}_H + D\mathcal{O}_L + E\mathcal{O}_p \\ \xi_3 &= -\mathcal{O}_L + F\mathcal{O}_p \\ \xi_4 &= \mathcal{O}_p \end{aligned} \tag{2}$$

will produce $\gamma_i := \gamma(\xi_i)$ satisfying the above hypotheses.

Mirror symmetry and CY-variations of Hodge structures

In this case, taking

$$\sigma_\infty(\alpha) := \lim_{q\to 0} \tilde{\sigma}(\alpha), \;\; \gamma_\infty(\xi) := \sigma_\infty\left(\hat{\Gamma}(X^\circ) \cup ch(\xi)\right),$$

we have $\gamma_i^{lim} = \gamma_\infty(\xi_i)$.

We now run this computation. Let $c(X^\circ) = 1 + a[L] + b[p]$ be the Chern class of X°. The Chern character is $ch(X^\circ) = 3 - a[L] + \frac{b}{2}[p]$ and the Todd class is $Td(X^\circ) = 1 + \frac{a}{12}[L]$, $\hat{\Gamma}(X^\circ) = 1 + \frac{a}{24}[L] - \frac{b\zeta(3)}{(2\pi i)^3}[p]$. This yields:

$$\gamma_3^{lim} = e_3 + Ae_2 + \left(-B + \frac{m}{2}A - \frac{a}{24}\right)e_1 + \Bigg(C - B + \frac{4m+a}{24}A - b\frac{\zeta(3}{(2\pi i}$$

$$\gamma_2^{lim} = e_2 + \left(-D + \frac{m}{2}\right)e_1 + \left(E - D + \frac{4m+a}{24}\right)e_0$$

$$\gamma_1^{lim} = e_1 + (F+1)e_0$$

$$\gamma_0^{lim} = e_0$$

Mirror symmetry and CY-variations of Hodge structures

Imposing the symplectic condition produces:

$$\Omega_{lim} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{a}{24} & -\frac{m}{2} & 1 & 0 \\ \frac{b\zeta(3)}{(2\pi i)^3} & \frac{a}{24} & 0 & 1 \end{pmatrix}. \quad (3)$$

To compute N (with these normalizations), we apply $\mathcal{O}(-H)\otimes$ to the ξ_i in $K_0^{num}(X^\circ)$; then

$$[T]_\gamma = [\mathcal{O}(-H)\otimes]_\xi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & m & 1 & 0 \\ -\frac{a+2m}{12} & m & 1 & 1 \end{pmatrix},$$

whereupon taking log gives

$$[N_{lim}]_{\gamma^{lim}} = [N]_\gamma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \frac{m}{2} & m & 0 & 0 \\ -\frac{a}{12} & \frac{m}{2} & -1 & 0 \end{pmatrix}.$$

Mirror symmetry and CY-variations of Hodge structures

The data required to compute N and Ω_{lim} for the complete intersection Calabi-Yau (CICY) examples from Doran-Morgan is displayed in the table:

X°	m	a	b
$\mathbb{P}^4[5]$	5	50	-200
$\mathbb{P}^5[2,4]$	8	56	-176
$\mathbb{P}^5[3,3]$	9	54	-144
$\mathbb{P}^6[2,2,3]$	12	60	-144
$\mathbb{P}^7[2,2,2,2]$	8	64	-128
$\mathbb{WP}^4_{1,1,1,2,5}[10]$	10	340	-2880
$\mathbb{WP}^4_{1,1,1,1,4}[8]$	8	176	-1184
$\mathbb{WP}^5_{1,1,2,2,3,3}[6,6]$	36	792	-4320
$\mathbb{WP}^5_{1,1,1,2,2,3}[4,6]$	24	384	-1872
$\mathbb{WP}^4_{1,1,1,1,2}[6]$	6	84	-408
$\mathbb{WP}^5_{1,1,1,1,1,3}[2,6]$	12	156	-768
$\mathbb{WP}^5_{1,1,1,1,2,2}[4,4]$	16	160	-576
$\mathbb{WP}^5_{1,1,1,1,1,2}[3,4]$	12	96	-312

The Arithmetic of the LG model of a certain class of threefolds

Next we turn our attention to the next result. The goal will be to use normal functions to give a 'motivic' meaning to constants arising in quantum differential equations associated to a certain class of Landau-Ginzburg models. Henceforward we will be mainly concerned with the Landau-Ginzburg models for a special class of threefolds, namely the ones whose associated local system is of rank three, with a single nontrivial involution exchanging two maximally unipotent monodromy points. More precisely, we will work with the varieties V_{12}, V_{16}, V_{18} and "R_1", where the first three are rank 1 Fanos appearing in the work of Golyshev and the latter is a rank 4 threefold with $-K^3 = 24$ (K the canonical divisor). The involutions for these LG models have essentially been described by Golyshev. In the presence of an involution, it is possible to move the degeneracy locus of a higher cycle from the fiber over 0 to its involute, a property which we use for the construction of the desired normal function.

The Arithmetic of the LG model of a certain class of threefolds

Definition (GOLYSHEV-2009)

Given a linear homogeneous recurrence R and two solutions $a_n, b_n \in \mathbb{Q}$ with $a_0 = 1, b_0 = 0, b_1 = 1$, if there is a L-function $L(x)$ and $c \in \mathbb{Q}^*$ such that:

$$\lim \frac{b_n}{a_n} = cL(x_0) \tag{4}$$

We say that yhe limit above is the Apéry constant of R.

Golyshev uses quantum recurrences of the threefolds V_{10}, V_{12}, V_{14}, V_{16}, V_{18} to find Apéry constants; his method is basically to use a result of Beukers for the rational cases and apply a different approach for the non-rational ones. In the course of the proof of his results, he also describes the involution we mentioned above.

The Arithmetic of the LG model of a certain class of threefolds

We will prove the following:

Theorem

Let X be a Fano threefold, in the special class described above. Then there is a higher normal function $\mathcal{N}$, arising from a family of motivic cohomology classes on the fibers of the LG model, such that the Apéry constant is equal to $\mathcal{N}(0)$.

We need this definition first:

Definition

For X_t a general $K3$ surface of the family induced by a Minkowski polynomial ϕ, let $X_t^* = X_t \cap (\mathbb{C}^*)^3$; then ϕ is tempered if the image of the higher Chow cycle $\xi_t := \langle x, y, z \rangle_{X_t^*} \in CH^3(X_t^*, 3)$ under all residue maps vanishes.

The Arithmetic of the LG model of a certain class of threefolds

Let X be one of the threefolds $V_{12}, V_{16}, V_{18}, R_1$. Associated to X is a Newton polytope Δ, and to the latter we associate a Minkowski polynomial ϕ. We have that ϕ is tempered, and the family of higher Chow cycles lifts to a class $[\Xi] \in CH^3(\mathcal{X}^\circ, 3)$, yielding by restriction a family of motivic cohomology classes $[\Xi_t] \in H^3_{\mathcal{M}}(X_t, \mathbb{Q}(3))$ on the Landau-Ginzburg model. The local system $\mathbb{V} = R^2_{tr}\pi_*\mathbb{Z}$ associated to the Landau-Ginzburg model of X has the following singular points:

- V_{12} : $t = 0, 17 \pm 12\sqrt{2}, \infty$
- V_{16} : $t = 0, 12 \pm 8\sqrt{2}, \infty$
- V_{18} : $t = 0, 9 \pm 6\sqrt{3}, \infty$
- R_1 : $t = 0, 4, 16, \infty$

The Arithmetic of the LG model of a certain class of threefolds

In each case, we have an involution $\iota(t) = \frac{M}{t}$, $(M = 1, \frac{1}{16}, \frac{-1}{27}, 64)$, exchanging say t_1 and t_2 with $0 < |t_1| < |t_2| < \infty$. The involution ι gives then a correspondence $I \in Z^2(\mathcal{X} \times \iota^*\mathcal{X})$ which gives a rational isomorphism between $\mathbb{V}$ and $\iota^*\mathbb{V}$.

Now let $\tilde{\Xi} := I^*\Xi \in H^3_{\mathcal{M}}(\mathcal{X}_\circ, \mathbb{Q}(3))$ be the pullback of the cycle, with fiberwise slices $\tilde{\Xi}_t$. If AJ is the Abel-Jacobi map as above, then

$$AJ^{3,3}([\tilde{\Xi}_t]) \in H^2(X_t, \mathbb{C}/\mathbb{Q}(3)). \tag{5}$$

Taking $\mathcal{R}_t$ to be any lift of this class to $H^2(X_t, \mathbb{C})$, and letting $\omega_t = \frac{1}{(2\pi i)^2} Res_{X_t}\left(\frac{\frac{dx_1}{x_1}\wedge\frac{dx_2}{x_2}\wedge\frac{dx_3}{x_3}}{1-t\phi}\right)$; we may define a normal function by:

$$\mathcal{N}(t) := \langle \mathcal{R}_t, \omega_t \rangle \tag{6}$$

The Arithmetic of the LG model of a certain class of threefolds

We have that:

$$D_{PF}(\mathcal{N}(t)) = kt, k \in \mathbb{Q}^*. \tag{7}$$

If $A(t) = \sum a_n t^n$ is the period sequence, then $B(t) = \sum b_n t^n = -\mathcal{N}(t) + A(t)\mathcal{N}(0)$ is another solution for the Picard-Fuchs equation, so that

$$\mathcal{N}(t) = \sum (a_n \mathcal{N}(0) - b_n) t^n.$$

Since the radii of convergence for the generating series of a_n and b_n are both $|t_1| < |t_2|$, while that of $a_n\mathcal{N}(0) - b_n$ is $|t_2|$, it follows that $\frac{b_n}{a_n} \to \mathcal{N}(0)$, which finishes the proof of the theorem.

The Arithmetic of the LG model of a certain class of threefolds

As a corollary we have:

Corollary

$\mathcal{N}(0)$ is (up to $\mathbb{Q}(3)$) a multiple of $\zeta(3)$.

The proof is a direct consequence of the following commutative diagram (After the work of Kerr-Lewis):

$$\begin{array}{ccc} H^3_{\mathcal{M}}(X_0, \mathbb{Q}(3)) & \xrightarrow{\cong} & K_5^{ind}(\mathbb{Q}) \\ \downarrow AJ^{3,3} & & \downarrow r_b \\ J^{3,3}(X_0) & \xrightarrow[\cong]{} & \frac{\mathbb{C}}{\mathbb{Q}(3)} \end{array} \tag{8}$$

Where the lower isomorphism is the pairing with ω_0 and r_b is the Borel regulator. The Abel-Jacobi map then reduces to the Borel regulator and by Borel's theorem it has to be multiple of $\zeta(3)$.

Surfaces with exceptional monodromy

There have been several constructions of family of varieties with exceptional monodromy group(Dettweiler-Reiter, Yun). In most cases, these constructions give Hodge structures with high weight. Nicholas Katz was the first to obtain Hodge structures with low weight(Hodge numbers not spread out) and geometric monodromy group G_2. In last part of this presentation I will describe Katz's construction and give a geometric proof that the geometric monodromy group of the family constructed by him is G_2.

Surfaces with exceptional monodromy

In his work, Katz describes 4 families, 3 of which have G_2 as geometric monodromy group. For the sake of simplicity, I will work with one of the 3 families, but the exact same approach applies to the remaining two in which G_2 occurs.

Let $\mathcal{E} \to \mathbb{P}^1 : y^2 = x(x-1)(x-z^2)$ be a rational elliptic surface with singular fibers at $z = -1, 0, 1, \infty$. For $t \neq 0, \pm\frac{2}{3\sqrt{3}}, \infty$, take a base change by:

$$E_t \to \mathbb{P}^1 : w^2 = tz(z-1)(z+1) + t^2 \tag{9}$$

The result is a family of elliptic surfaces $X_t \to E_t$ with 7 singular fibers on each surface, as described below:

$$\begin{array}{ccc} \mathcal{X} & \hookleftarrow & X_t \\ \downarrow \pi & & \downarrow \pi_t \\ \mathcal{E} & \hookleftarrow & E_t \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \hookleftarrow & \{t\} \end{array} \tag{10}$$

Surfaces with exceptional monodromy

Proposition

For each X_t we have $dim(H^2_{tr}(X_t)) = 7$.

We now describe a particular choice of 7-dimensional basis of 2-cycles that we will use henceforward. First, consider the 1-cycles $\alpha, \beta, \gamma_{-1}, \gamma_0, \gamma_1$ over each E_t, as described in figure 1. Denote by δ_1, δ_2 the basis for the local system over each point of E_t, with $\delta_1 \cdot \delta_2 = 1$.

Surfaces with exceptional monodromy

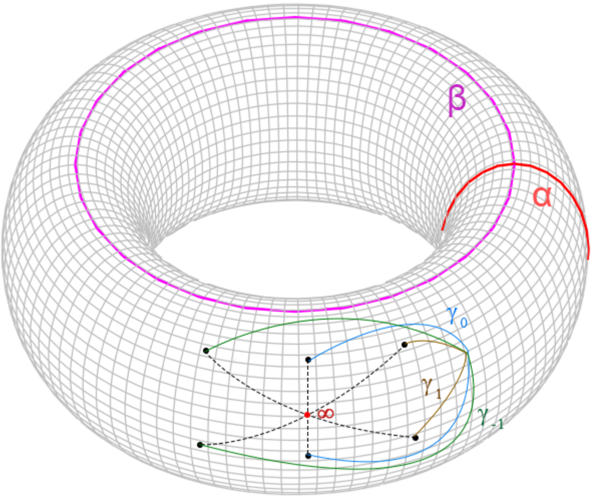

Figure: 1-cycles over the Base E_t

Surfaces with exceptional monodromy

The local monodromies around $-1, 0, 1$ for the family $\mathcal{E} \to \mathbb{P}^1$: $y^2 = x(x-1)(x-z^2)$ are:

$$\begin{aligned} \tilde{T_{-1}} &= \begin{pmatrix} -3 & 8 \\ -2 & 5 \end{pmatrix} \\ \tilde{T}_0 &= \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} \\ \tilde{T}_1 &= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \end{aligned} \tag{11}$$

The vanishing cyle at each singular point is then:

- $2\delta_1 + \delta_2$ at -1
- δ_1 at 0
- δ_2 at 1

Surfaces with exceptional monodromy

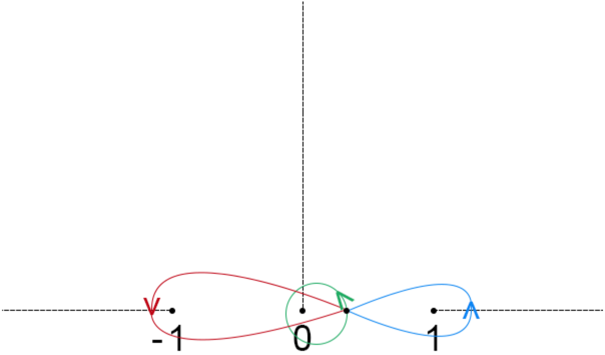

Figure: Cycles enclosing -1,0 and 1 in $\mathbb{P}^1$ minus the cuts.

Surfaces with exceptional monodromy

Set $\eta_1 = \delta_2$ and $\eta_2 = 2\delta_1 + \delta_2$, so $\eta_1 \cdot \eta_2 = -2$ and the vanishing cycle at 0 is precisely $\frac{1}{2}(\eta_1 + \eta_2)$. We use henceforward the notation $a \times b$ to denote the 2-cycle on X_t obtained by taking the 1-cycle a on a fiber of π_t and continuing it along the 1-cycle b.

Surfaces with exceptional monodromy

Now that our notation is stablished we procced with the definition of our 7-dimensional basis of $H^2_{tr}(X_t)$:

$$\begin{aligned} A_1 &= \eta_1 \times \alpha & C_{-1} &= \eta_2 \times \gamma_{-1} \\ A_2 &= \eta_2 \times \alpha & C_0 &= \frac{1}{2}(\eta_1 + \eta_2) \times \gamma_0 \\ B_1 &= \eta_1 \times \beta & C_1 &= \eta_1 \times \gamma_1 \\ B_2 &= \eta_2 \times \beta \end{aligned} \tag{12}$$

Surfaces with exceptional monodromy

Note that, A_1, A_2, B_1, B_2 are trivially transcendental, the same is not true for the C_i. The reason is that the C_i may–in fact they do–contain algebraic cycles resulting from classes of singular fibers. To overcome this, we have to "add" enough cycles in order to make all C_i transcendental.

Surfaces with exceptional monodromy

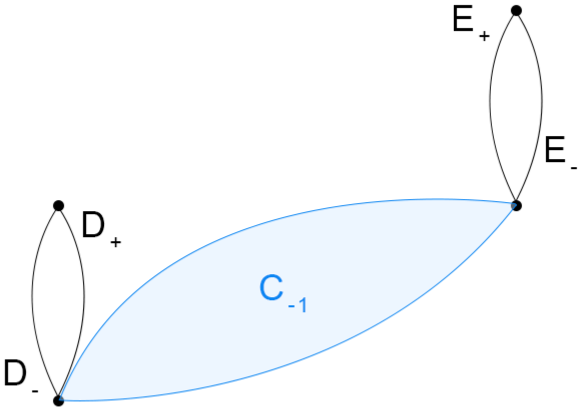

Figure: The 2-cycle C_{-1}

Surfaces with exceptional monodromy

Now, we eliminate the algebraic components of C_{-1}. Set:

$$\widetilde{C}_{-1} := C_{-1} + aD_{-} + bD_{+} + cE_{-} + dE_{+} \tag{13}$$

After imposing the transcendency conditions we get:

$$\widetilde{C}_{-1} = C_{-1} + \frac{1}{2}D_{-} + -\frac{1}{2}E_{-} \tag{14}$$

By following the exact same reasoning, we deduce that:

$$\widetilde{C}_{1} = C_{1} + \frac{1}{2}G_{-} + -\frac{1}{2}H_{-} \tag{15}$$

where G_{-} and H_{-} are the components of the singular fibers of the endpoints.

Surfaces with exceptional monodromy

Now we address C_0, consider the figure 4. Following the idea above, we set:

$$\widetilde{C}_0 = C_0 + aL_1 + bL_2 + cL_3 - dF_1 - eF_2 - fF_3 \qquad (16)$$

We again solve the system of equations required for transcendency to obtain:

$$\widetilde{C}_0 = C_0 + \frac{3}{4}L_1 + \frac{1}{2}L_2 + \frac{1}{4}L_3 - \frac{3}{4}F_1 - \frac{1}{2}F_2 - \frac{1}{4}F_3 \qquad (17)$$

Surfaces with exceptional monodromy

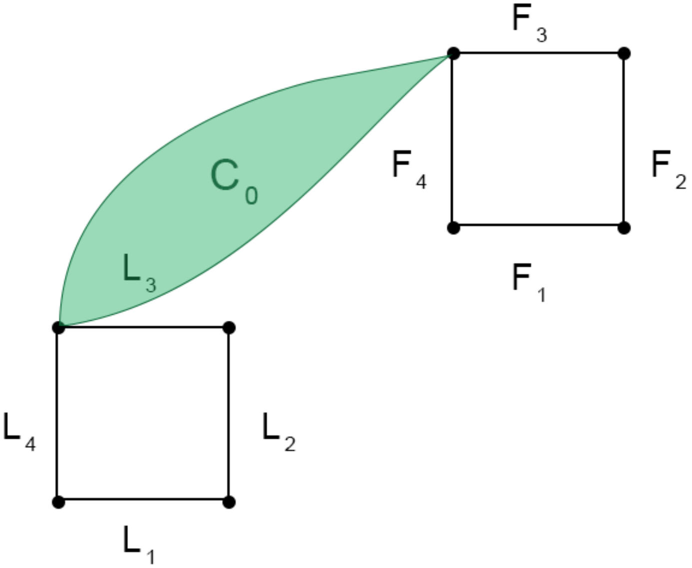

Figure: The 2-cycle C

Surfaces with exceptional monodromy

Denote by V the space generated by the transcendental cycles $(A_1, A_2, B_1, B_2, \widetilde{C_{-1}}, \widetilde{C_0}, \widetilde{C_1})$. The intersection matrix is:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & -1 \\ 0 & 0 & 0 & 0 & -2 & -1 & -1 \end{bmatrix}$$

We now compute the monodromies matrices at the singular points $t_- := \frac{-2}{3\sqrt{3}}, 0, t_+ := \frac{2}{3\sqrt{3}}, \infty$, restricted to the vector space V.

Surfaces with exceptional monodromy

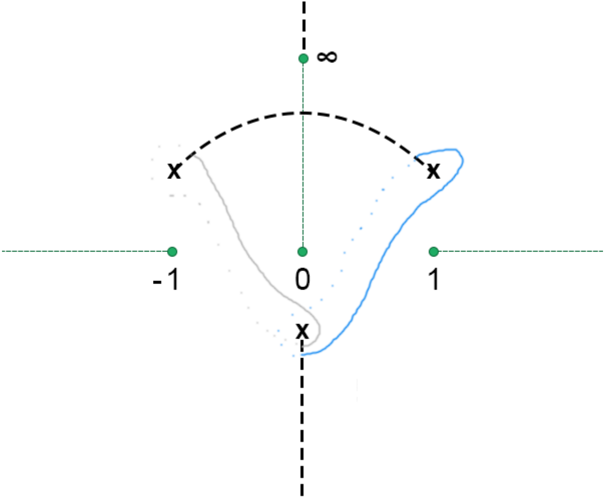

Figure: The 1-cycles α and β over the Elliptic curve E_t

Surfaces with exceptional monodromy

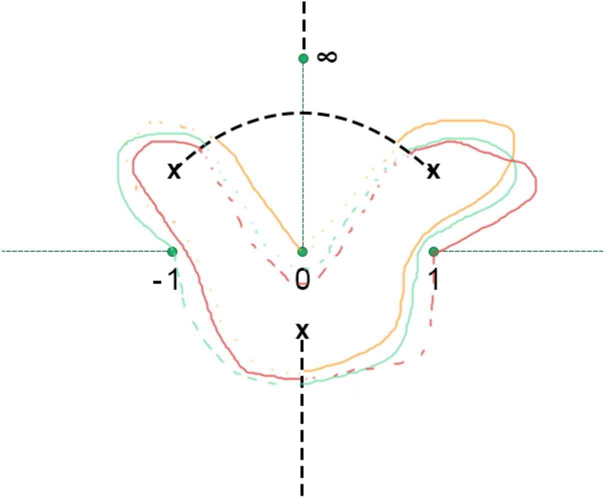

Figure: The 1-cycles γ_{-1},γ_0 and γ_1 over the Elliptic curve E_t.

Surfaces with exceptional monodromy

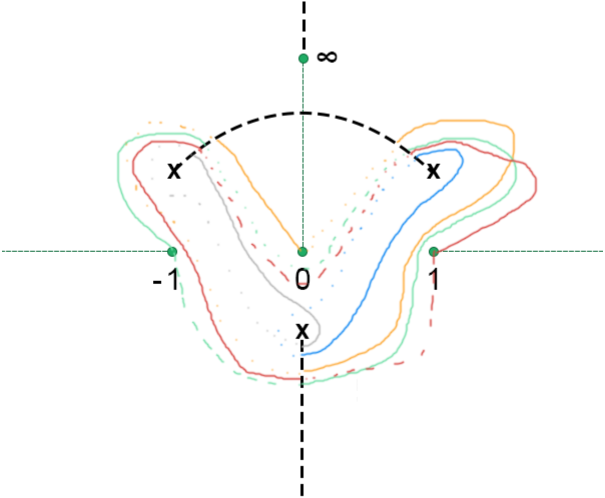

Figure: The 1-cycles over the Elliptic curve E_t

Surfaces with exceptional monodromy

When $t \to t_{\pm}$, we have a nodal degeneration on E_t. It's straightforward to conclude that in this case, the $\widetilde{C}_i$ remain unchanged. Moreover, the A_i, B_i change according to the Picard-Lefschetz formula, hence:

$$M_+ = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, M_- = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Surfaces with exceptional monodromy

The situation when $t \to 0$ is more subtle. If one looks at figure 7, the endpoints of the cuts behave roughly as $-1 - \frac{t}{2}, t$ and $1 - \frac{t}{2}$, therefore when t go through a path around 0, the endpoints will certain move, but this time not in a nice way as they did in the case above, they will instead make the γ_i cycles cross each other and also α and β.

Surfaces with exceptional monodromy

This is the α after we apply monodromy:

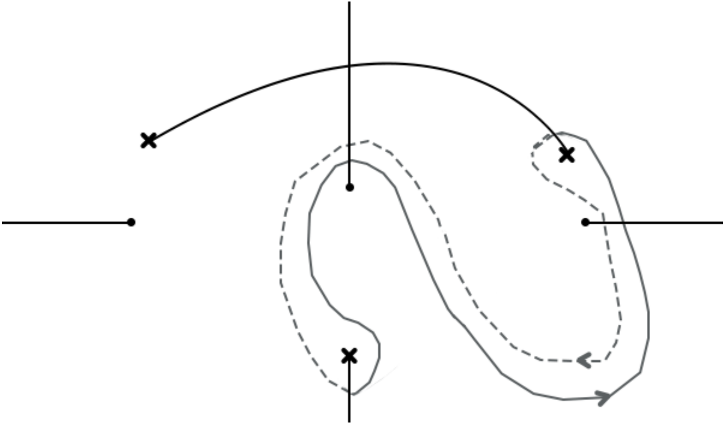

Figure: $\widetilde{\alpha}$, the resulting cycle after monodromy

Surfaces with exceptional monodromy

This is the γ_0 after we apply monodromy:

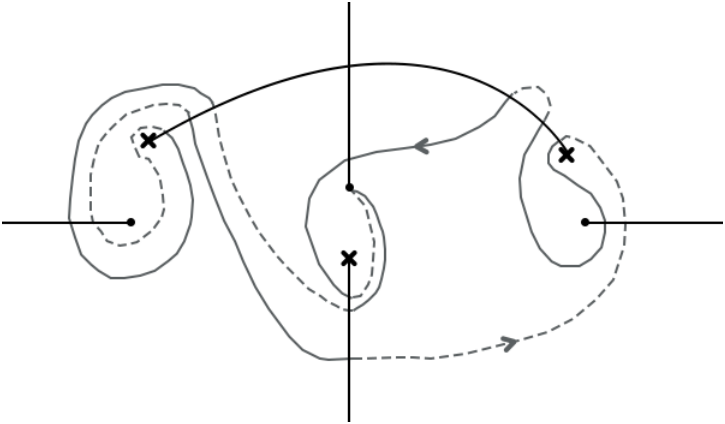

Figure: $\widetilde{\gamma_0}$, the resulting cycle after monodromy

Surfaces with exceptional monodromy

Using the expression for the local monodromies, we arrive at:

$$M_0 = \begin{bmatrix} 1 & 2 & -2 & -2 & 2 & -1 & -2 \\ -2 & -3 & 6 & 2 & -4 & 3 & 6 \\ 2 & 6 & -3 & -2 & 6 & -3 & -4 \\ -2 & -2 & 2 & 1 & -2 & 1 & 2 \\ 0 & 0 & -4 & 0 & 1 & -2 & -4 \\ -4 & -4 & 4 & 4 & -4 & 1 & 4 \\ 0 & -4 & 0 & 0 & -4 & 2 & 1 \end{bmatrix}$$

Surfaces with exceptional monodromy

Since we can rearrange the loops around $t_-, 0, t_+, \infty$ so that their product is the identity, we naturally get the expression for M_∞ as the inverse of the prodcut $M_- \cdot M_0 \cdot M_+$, leading to:

$$M_\infty = \begin{bmatrix} 0 & -4 & 1 & 0 & -4 & 2 & 2 \\ 4 & 0 & 4 & 1 & -2 & 2 & 4 \\ -1 & 4 & -3 & -2 & 6 & -3 & -4 \\ 0 & -1 & 2 & 1 & -2 & 1 & 2 \\ -4 & 0 & -4 & 0 & 1 & -2 & -4 \\ 0 & 0 & 4 & 4 & -4 & 1 & 4 \\ 0 & -4 & 0 & 0 & -4 & 2 & 1 \end{bmatrix}$$

We can easily check that M_-, M_+, M_0, M_∞ preserve the intersection form Q, hence the subgroup $\Gamma \subset GL(7)$ they generate is in fact inside $SO(3,4)$.

Surfaces with exceptional monodromy

We now describe the logarithm of the M_i. A quick computation shows that M_0 is semi-simple, hence the unipotent part of M_0 is the identity, so $N_0 = 0$. The remaining monodromies do have non trivial logarithms: M_+, M_- are actually unipotent and M_∞ is the only non-unipotent. We can easily check that M_∞^3 is unipotent though. If $M_\infty = M_s \cdot M_u$ is the Jordan-Chevalley decomposition and I is the 7x7 identity matrix, then:

$$\begin{aligned} N_+ &= M_+ - I \\ N_- &= M_- - I \\ N_\infty &:= log(M_u) = \frac{1}{3} log(M_\infty^3) \end{aligned} \tag{18}$$

Surfaces with exceptional monodromy

We have the following theorem:

Theorem

The log-monodromies N_+, N_-, N_∞ generate $\mathfrak{g}_2$, therefore the geometric monodromy group for the Katz family is G_2.

Proof: Consider the elements:

$$
\begin{aligned}
Y_1 &= [N_-, N_+] & Y_8 &= [Y_5, Y_6] \\
Y_2 &= [N_-, N_\infty] & Y_9 &= [N_\infty, Y_5] \\
Y_3 &= [N_+, N_\infty] & Y_{10} &= [N_\infty, Y_9] \\
Y_4 &= [Y_1, Y_2] & Y_{11} &= [N_\infty, Y_{10}] \\
Y_5 &= [Y_1, Y_3] & Y_{12} &= [N_+, Y_{11}] \\
Y_6 &= [Y_2, Y_3] & Y_{13} &= [N_\infty, Y_{12}] \\
Y_7 &= [Y_2, Y_6] & Y_{14} &= [N_-, Y_{13}]
\end{aligned}
\tag{19}
$$

Surfaces with exceptional monodromy

A quick computation shows that the elements N_-, N_+, Y_1, Y_4, Y_5, Y_6, Y_7, Y_8, Y_9, Y_{10}, Y_{11}, Y_{12}, Y_{13}, Y_{14} are linearly independent over $\mathbb{Q}$. Now define $t_1 := Y_1$ and $t_2 := [Y_4, Y_5]$, a direct computation gives us that $[t_1, t_2] = 0$, moreover they both are diagonalizable. Let $ad(.)$ denotes the adjoint representation, if we act through $ad(t_i)$, $i = 1, 2$, on $\mathfrak{g}$, we get 14 linearly independent (simultaneous for t_1, t_2) eigenvectors with 1-dimensional eigenspaces, moreover we have:

- 1 with eigenvalue -2
- 4 with eigenvalue -1
- 4 with eigenvalue 0
- 4 with eigenvalue 1
- 1 with eigenvalue 2

Which are in 1-1 correspondence with the roots of $\mathfrak{g}_2$, therefore $\mathfrak{h} := \langle t_1, t_2 \rangle$ is a Cartan subalgebra and $\mathfrak{g} = \mathfrak{g}_2$.

$\square$

Thanks!